

# Models in Boundary Quantum Field Theory Associated with Lattices and Loop Group Models

MARCEL BISCHOFF

Dipartimento di Matematica, Università di Roma “Tor Vergata”

Via della Ricerca Scientifica, 1, I-00133 Roma, Italy

E-mail address: `bischoff@mat.uniroma2.it`

**ABSTRACT.** In this article we give new examples of models in boundary quantum field theory, i.e. local time-translation covariant nets of von Neumann algebras, using a recent construction of Longo and Witten, which uses a local conformal net  $\mathcal{A}$  on the real line together with an element of a unitary semigroup associated with  $\mathcal{A}$ . Namely, we compute elements of this semigroup coming from Hölder continuous symmetric inner functions for a family of (completely rational) conformal nets which can be obtained by starting with nets of real subspaces, passing to its second quantization nets and taking local extensions of the former. This family is precisely the family of conformal nets associated with lattices, which as we show contains as a special case the level 1 loop group nets of simply connected, simply laced groups. Further examples come from the loop group net of  $\mathrm{Spin}(n)$  at level 2 using the orbifold construction.

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## 1. INTRODUCTION

In the operator algebraic approach to quantum field theory (QFT) one studies nets of operator algebras (e.g. von Neumann algebras) that assign to a space-time region the algebra of observables localized in it. These nets are asked to fulfill certain axioms coming from basic physical principles; we mention as examples the locality principle—which asks that the algebras assigned to causally disjoint regions should commute (local nets)—and the covariant assignment with respect to some “symmetry group” of the space-time (for a general introduction on this subject we refer to the textbook [Haa96]).

In this approach also conformal quantum field theory (CQFT) have been treated by considering nets on two dimensional Minkowski space and its chiral parts, which can be regarded as nets on the real line or as nets on the circle. Besides CQFT on the full Minkowski space also boundary conformal quantum field theory (BCFT) on the Minkowski half-plane  $x > 0$  is described in the algebraic approach. More

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precisely, in the paper [LR04] Longo and Rehren associate with a local conformal net  $\mathcal{A}$  on the real line a local conformal boundary net  $\mathcal{A}_+$  on Minkowski half-plane and obtain more general boundary nets which are extending  $\mathcal{A}_+$ .

Lately, in [LW10] Longo and Witten give a framework to construct models in boundary quantum field theory (BQFT) by investigating into local nets on the Minkowski half-plane, which are in general only time-translation covariant and can be considered as a deformation of the net  $\mathcal{A}_+$ . Specifically, the construction starts with a conformal net  $\mathcal{A}$  on the real line together with an element  $V$  of a unitary semigroup  $\mathcal{E}(\mathcal{A})$  associated with  $\mathcal{A}$  to construct a net on Minkowski half-plane, where the special case  $V = 1$  is the net  $\mathcal{A}_+$ . The search for new models is basically given by the construction of elements of the semigroup  $\mathcal{E}(\mathcal{A})$  for a given conformal net  $\mathcal{A}$ . Further in [LR11] Longo and Rehren investigate in BQFT on the Lorentz hyperboloid using a similar semigroup, nonetheless we will concentrate in this paper on BQFT on Minkowski half-plane. In this framework, an interesting class of conformal nets to consider are the completely rational conformal nets [KLM01] just having finite number of sectors (equivalence classes of irreducible representations) with each one having only finite statistics and their representation theory giving rise to modular tensor categories.

In this work the main goal is to construct elements of the semigroup  $\mathcal{E}(\mathcal{A})$  for the loop group models (which are “in general” expected to fulfill complete rationality) and hence give new models of BQFT. The loop group models are conformal nets coming from (projective) positive energy representations of loop groups. The cocycles of the projective positive energy representation of loop groups are classified by their level and the vacuum representation of each level yields a conformal net. In the case of simply laced Lie groups the level 1 representation is the basic representation and all higher level representations are contained in tensor products [GF93]; as a result the higher level loop group models are contained as subnets in the tensor product of the level 1 loop group net. So the first important step is to construct semigroup elements for the level 1 loop group net. This can be obtained as a subnet of a free Fermionic net or as an extension of a free Bosonic net (as we will show here); by free nets we mean second quantization nets using the CCR or CAR algebra of a net of real subspaces in the Bose and Fermi case, respectively. For these free nets the semigroup elements being second quantization unitaries are characterized in [LW10].

We look into extensions of free Bosons, namely the family of conformal nets associated with lattices and show that this family indeed contains the level 1 loop group models of simply laced groups as a special case. This can be regarded as an algebraic version of the Frenkel–Kac or Frenkel–Kac–Segal construction, which says that the lattice vertex operator algebras of simply laced root lattices correspond to the level 1 Kac–Moody vertex operator algebras (cf. Theorem 5.6 [Kac98]). Furthermore these family consists only of completely rational conformal nets as shown by Dong and Xu in [DX06].

In Section 2 we give some basic preliminaries on standard subspaces, its associated modular theory and semigroups of standard pairs.

In Section 3 we review the construction of the conformal nets under investigation starting with a net of standard subspaces in the spirit of [BGL02, Lon08b] and obtaining a net of von Neumann algebras by second quantization, which describes Abelian currents on the circle. Their local extensions by even lattices are shown to be the conformal nets associated with lattices constructed in [DX06] using positive energy representations of the loop group of the torus related to the lattice. We show that this family indeed contains as a special case the loop group nets at level 1 of simply laced groups.

In Section 4 we bring about a family of local nets on Minkowski half-plane associated with each step of the former construction of chiral models. The important step is the extension of the semigroup elements obtained by second quantization to the local extension by a lattice. We give criterion when such elements extend and also look into restriction to subnets. A further family of examples of semigroup elements and therefore models in BQFT are calculated for the loop group nets of  $\text{Spin}(n)$  at level 2 using the orbifold construction.

## 2. PRELIMINARIES ON STANDARD SUBSPACES

In this section we give some basic preliminaries on standard subspaces, its associated modular theory and semigroups of standard pairs.

**2.1. Standard Subspaces.** We repeat some basic facts (for details see [Lon08b]) on standard subspaces. Let  $\mathcal{H} \equiv (\mathcal{H}, (\cdot, \cdot))$  be a Hilbert space and let  $H \subset \mathcal{H}$  be a real subspace. We denote by  $H' = \{x \in \mathcal{H} : \text{Im}(x, H) = 0\}$  the *symplectic complement*, which is closed. In particular it is  $H'' = \overline{H}$ . A closed real subspace  $H$  is called *cyclic* if  $\overline{H} + i\overline{H} = \mathcal{H}$  and is called *separating* if  $H \cap iH = \{0\}$ . So a closed real subspace  $H$  is separating or cyclic if and only if its symplectic complement  $H'$  is cyclic or separating, respectively. A cyclic and separating subspace  $H$  is called *standard*; clearly  $H$  is standard if and only if  $H'$  is standard. We denote the set of all standard subspaces of  $\mathcal{H}$  by  $\text{Std}(\mathcal{H})$ . To a standard subspace we relate a pair  $(J_H, \Delta_H)$ , where  $(\Delta_H^{it})_{t \in \mathbb{R}}$  is a unitary one-parameter group called the *modular unitaries* and an antiunitary involution  $J_H$  called *modular conjugation*. Both are defined by the polar decomposition of the densely defined, closed, antilinear involutive (i.e.  $S_H^2 \subset \text{id}_{\mathcal{H}}$ ) operator  $S_H = J\Delta_H^{1/2}$  with domain  $H + iH$  defined by  $x + iy \mapsto x - iy$  for  $x, y \in H$ . A (simpler) real subspace version of the Tomita-Takesaki theorem gives:

$$JH = H', \quad \Delta_H^{it}H = H \quad (t \in \mathbb{R}).$$

We note that there is a useful bijective correspondence between  $\text{Std}(\mathcal{H})$  and the set of densely defined, closed, antilinear involutions  $S$  on  $\mathcal{H}$ , given by the map  $H \mapsto S_H$  as above, with inverse map associating with such an involution  $S$  the standard subspace  $H_S = \{x \in \text{Dom}(S) : Sx = x\} = \ker(1 - S)$ .

## 2.2. Semigroup Associated with Standard Pairs.

**Definition.** Let  $H$  be a standard subspace of a Hilbert space  $\mathcal{H}$  and let us assume that there exists a one-parameter group  $T(t) = e^{itP}$  on  $\mathcal{H}$  such that:

- $T(t)H \subset H$  for all  $t \geq 0$ ,
- $P > 0$ .

Then we call the pair  $(H, T)$  a *standard pair*. It is called *non-degenerated* if the kernel of  $P$  is  $\{0\}$ .

It holds a one-particle version of Borchers Theorem with some implications:

**Theorem 2.1** ([LW10, Theorem 2.2]). *Let  $(H, T)$  be a non-degenerate standard pair.*

- (1) *Then it holds for all  $t, s \in \mathbb{R}$ :*

$$\Delta_H^{is}T(t)\Delta_H^{-is} = T(e^{2\pi s}t), \quad JT(t)J = T(-t),$$

*where  $\Delta_H^{it}$  and  $J$  are the modular unitaries and conjugation, respectively, associated with the standard space  $H$ , i.e.  $JH = H'$  and  $\Delta_H^{it}H = H$ .*

- (2)  *$(H, T)$  yields a unitary positive energy representation of the translation-dilation group of  $\mathbb{R}$  also called the  $ax + b$  group, by associating with  $x \mapsto e^{-2\pi s}x + t$  the unitary element  $T(t)\Delta_H^{is}$ .*

- (3) *There is a unique irreducible standard pair and each standard pair is a multiple of this unique standard pair.*

**Definition.** Let  $(H, T)$  be a standard pair on  $\mathcal{H}$ . The semigroup of unitaries  $V$  of  $\mathcal{H}$  commuting with  $T$  such that  $VH \subset H$  is denoted by  $\mathcal{E}(H, T) = \mathcal{E}(H)$ .

The elements of  $\mathcal{E}(H)$  are characterized in [LW10]. We first state the case of the irreducible standard pair, where the semigroup  $\mathcal{E}(H_0)$  can be identified with a semigroup of certain “symmetric inner functions”.

**Definition.** We denote by  $\mathcal{S}$  the set of all complex Borel functions  $\varphi : \mathbb{R} \rightarrow \mathbb{C}$  which are boundary values of a bounded analytic function on  $\mathbb{R} + i\mathbb{R}_+$ , which are symmetric, i.e.  $\overline{\varphi(p)} = \varphi(-p)$  and inner, i.e.  $|\varphi(p)| = 1$  for almost all  $p \geq 0$ .

**Theorem 2.2** ([LW10, Corollary 2.4]). *Let  $(H_0, T_0)$  be the unique irreducible standard pair then  $V \in \mathcal{E}(H)$  if and only if  $V = \varphi(P)$  for some  $\varphi \in \mathcal{S}$ .*

In the reducible case the semigroup  $\mathcal{E}(H)$  consists of matrices of similar functions and the condition  $|f(p)| = 1$  is generalized to unitarity of the matrix.

**Remark 2.3.** *Let  $(H, T)$  be a non-zero, non-degenerated standard pair on a Hilbert space  $\mathcal{H}$ . Then it can be decomposed as a direct sum of the unique irreducible standard pair. Let*

$$\mathcal{H} = \bigoplus_i \mathcal{H}_i \quad H = \bigoplus_i H_i \quad T = \bigoplus_i T_i$$

*be such a finite or infite decomposition, where each  $(H_i, T_i)$  is a standard pair in  $\mathcal{H}_i$  and can be identified with the unique irreducible standard pair  $(H_0, T_0)$  with generator  $P_0$ .*

**Definition.** For  $n \in \mathbb{N} \cup \{\infty\}$  we denote by  $\mathcal{S}^{(n)}$  the set of matrices  $(\varphi_{hk})_{1 \leq h, k \leq n}$  where  $\varphi_{hk} : \mathbb{R} \rightarrow \mathbb{C}$  are complex Borel functions which are boundary values of a bounded analytic function on  $\mathbb{R} + i\mathbb{R}_+$  such that  $\varphi_{hk}(p)$  is a unitary matrix for almost all  $p$ , which is symmetric, i.e.  $\overline{\varphi_{hk}(p)} = \varphi_{hk}(-p)$ .

**Theorem 2.4** ([LW10, Theorem 2.6]). *Let  $H$  be like in Remark 2.3. Then  $V \in \mathcal{E}(H)$  if and only if it is a  $n \times n$  matrix  $(V_{hk})$  with entries in  $\mathcal{B}(\mathcal{H})$  such that  $V_{hk} = \varphi_{hk}(P_0)$  for some  $(\varphi_{hk}) \in \mathcal{S}^{(n)}$ .*

### 3. CONFORMAL FIELD THEORY – CONFORMAL NETS

In this section we are interested in local Möbius covariant nets (conformal nets). These are nets on the circle (or the real line), which physically describe the chiral part of the algebra of observables of a 2D QFT, where the real line (circle) is then identified with (the compactification) of one of the lightrays.

**3.1. Nets of Standard Subspaces.** Before describing nets of von Neumann algebras we want to go a step back and give some details on nets of real subspaces of a Hilbert space  $\mathcal{H}_0$ , whose “second quantization” leads to nets of von Neumann algebras, the so called second quantization nets. In analogy to the “free field construction” from Wigner particles the Hilbert space  $\mathcal{H}_0$  will be called the “one-particle space”. See for example [BGL02] for a general construction of free Bosons using this techniques on more general space-times<sup>1</sup> and [Lon08b] for such nets on the circle.

<sup>1</sup>In our case the “space-time” is the circle and the “wedges” correspond to open non-empty nowhere dense intervals

We will identify the one-point compactification  $\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$  of the real line with the circle  $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$  by the Caley map

$$C : \overline{\mathbb{R}} \hookrightarrow \mathbb{S}^1, \quad x \mapsto -\frac{x-i}{x+i} \iff x = C^{-1}(z) = -i \frac{z-1}{z+1}.$$

Our symmetry group should be the *group of Möbius transformations*  $\text{Möb}$  of the circle and can be identified with either  $\text{PSL}(2, \mathbb{R})$  or  $\text{PSU}(1, 1)$ , which act naturally on the compactified real line  $\overline{\mathbb{R}}$  and the circle  $\mathbb{S}^1$ , respectively. The Möbius group is generated by the following three one-parameter subgroups: the *rotations*  $R(\theta)z = e^{i\theta}z$ , which are easier in the circle picture; the *translations*  $\tau(t)x = x + t$  and *dilations*  $\delta(s)x = e^s x$  for  $x \in \overline{\mathbb{R}}$  which are both easier in the real line picture. We add the orientation reversing *reflection*  $rz = \bar{z}$  with  $z \in \mathbb{S}^1$  to  $\text{Möb}$  and denote the obtained group by  $\text{Möb}_{\pm} = \text{Möb} \rtimes_{\text{Ad } r} \mathbb{Z}_2$ . For  $z \in \mathbb{S}^1$  we sometimes write  $z = e^{i\theta}$  and note that  $x \equiv C^{-1}(e^{i\theta}) = \tan(\theta/2)$ .

A connected, non-empty, nowhere dense interval  $I \subset \mathbb{S}^1$  is called *proper* and we denote by  $\mathcal{I}$  the by inclusions partial ordered set of all proper intervals. For  $I \in \mathcal{I}$  we denote by  $I'$  the interior of  $\mathbb{S}^1 \setminus I$  which is clearly in  $\mathcal{I}$  and note that  $\text{Möb}$  acts transitive on  $\mathcal{I}$ .

**Definition.** A *strongly continuous unitary representation of Möb* (or a subgroup containing the rotations) on a Hilbert space  $\mathcal{H}$  is called *positive energy representation* if the generator  $L_0$  of the one-parameter subgroup of rotations  $U(R(\theta)) = e^{i\theta L_0}$  has positive spectrum. The representation is called *non-degenerate* if it does not contain the trivial representation.

**Remark 3.1** ([Lon08b, Theorem 2.10]). A *unitary positive energy representation of Möb* extends to a (anti-) unitary representation of  $\text{Möb}_{\pm}$  on the same Hilbert space and the extension is unique up to unitary equivalence.

**Definition.** A *local Möbius covariant net of standard subspaces of  $\mathcal{H}$*  is a family of standard subspaces  $H(I) \subset \mathcal{H}$  indexed by  $I \in \mathcal{I}$  such that the following properties hold:

- A. Isotony.**  $I_1 \subset I_2$  implies  $H(I_1) \subset H(I_2)$ .
- B. Locality.**  $I_1 \cap I_2 = \emptyset$  implies  $H(I_1) \subset H(I_2)'$ .
- C. Möbius covariance.** There is a positive energy representation of  $\text{Möb}$  on  $\mathcal{H}$  such that  $U(g)H(I) = H(gI)$  for all  $g \in \text{Möb}$  and  $I \in \mathcal{I}$ .
- D. Irreducibility.**  $U$  is non-degenerate, i.e. does not contain the trivial representation.

Given a positive energy representation  $U$  of  $\text{Möb}$  on  $\mathcal{H}$  we can construct a local Möbius covariant net of standard subspaces as follows: we define the unitary one-parameter group  $\Delta^{it} = U(\delta(-2\pi t))$  where  $\delta(t)x = e^t x$  are the dilations and the antiunitary involution  $J = U(r)$  (where we use that  $U$  extends to a representation of  $\text{Möb}_{\pm}$ ) and define the densely defined, closed, antilinear involution  $S = J\Delta^{1/2}$ . We denote by  $I_0$  the interval corresponding to the upper circle or equivalently  $(0, \infty)$ . Then we set  $H(I_0) \equiv H(0, \infty) = \{x \in \mathcal{H} : Sx = x\}$  to be the standard subspace associated with  $S$  and for general  $\mathcal{I} \ni I = gI_0$  we set  $H(I) = U(g)H(0, \infty)$ , which does not depend on the choice of  $g \in \text{Möb}$ . All local Möbius covariant nets of standard subspaces are obtained in this way [Lon08b].

For later use we make the construction of a family indexed by  $n \in \mathbb{N}$  of local Möbius covariant nets of real subspaces—namely the net coming from  $n$  copies of the lowest weight 1 positive energy representation (cf. [Lon08b]) of  $\text{Möb}$ —more explicit. Therefore let  $F$  be a  $n$ -dimensional Euclidean space with scalar product  $\langle \cdot, \cdot \rangle$ . Let us define  $\mathcal{H}_{0,F} = \mathcal{H}_0 \otimes_{\mathbb{R}} F \cong \bigoplus_{i=1}^n \mathcal{H}_0$  which is in particular isomorphic to the  $n$ -fold direct sum of the unique irreducible positive energy lowest weight

representation of Möb with lowest weight 1 denoted by  $(U_0, \mathcal{H}_0)$ . We denote by  $U_{0,F}$  the unitary representation of the Möb on  $\mathcal{H}_{0,F}$ . It can explicitly be constructed as follows. Let  $\mathbb{L}F = C^\infty(\mathbb{S}^1, F) \cong C^\infty(\mathbb{S}^1, \mathbb{R}) \otimes_{\mathbb{R}} F$  the set of all smooth maps (loops) from the circle  $\mathbb{S}^1$  in  $F$ . Because  $f \in \mathbb{L}F$  is periodic it can be written as a Fourier series

$$f(\theta) = \sum_{k \in \mathbb{Z}} \hat{f}_k e^{ik\theta}, \quad \hat{f}_k = \int_0^{2\pi} e^{-ik\theta} f(\theta) \frac{d\theta}{2\pi}$$

with Fourier coefficients  $\hat{f}_k = \overline{\hat{f}_{-k}}$  in the complexified space  $F_{\mathbb{C}} := F \otimes_{\mathbb{R}} \mathbb{C}$ . We introduce a semi-norm

$$\|f\|^2 = \sum_{k=1}^{\infty} k \cdot \|\hat{f}_k\|_{F_{\mathbb{C}}}^2$$

and a complex structure, i.e. an isometry  $\mathcal{J}$  w.r.t.  $\|\cdot\|$  satisfying  $\mathcal{J}^2 = -1$ , by

$$\mathcal{J} : \hat{f}_k \mapsto -i \operatorname{sign}(k) \hat{f}_k$$

and finally we get the Hilbert space  $\mathcal{H}_{F,0}$  by completion with respect to the norm  $\|\cdot\|$

$$\mathcal{H}_{0,F} = \overline{\mathbb{L}F/F}^{\|\cdot\|},$$

where  $F$  is identified with the constant functions. The scalar product  $(\cdot, \cdot)$  can be obtained by polarization and the unitary action of Möb is induced by the action on  $\mathbb{L}F$ , namely

$$U(g) : f \mapsto g_* f : (g_* f)(\theta) = f(g^{-1}(\theta)).$$

Let  $f \in \mathbb{L}F$ . If no confusion arises we denote also its image  $[f] \in \mathcal{H}_{0,F}$  of the inclusion  $\iota_F : \mathbb{L}F \rightarrow \mathcal{H}_{0,F}$  by  $f$ . On  $\mathbb{L}F$  the sesquilinear form coming from the scalar product is given explicitly by

$$\omega(f, g) := \operatorname{Im}(f, g) = \frac{-i}{2} \sum_{k \in \mathbb{Z}} k \langle \hat{f}_{-k}, \hat{g}_k \rangle = \frac{1}{2} \int_0^{2\pi} \langle f'(\theta), g(\theta) \rangle \frac{d\theta}{2\pi} =: \frac{1}{2} \int \langle f', g \rangle.$$

For  $I \in \mathcal{I}$  we denote by  $H_F(I)$  the closure subspace of functions with support in  $I$ . The family  $\{H_F(I)\}_{I \in \mathcal{I}}$  is a local Möbius covariant net of standard subspaces. Indeed because  $U$  acts geometrical, and in particular  $U(\delta(t))$  is the modular group of the abstract construction and leaves  $H_F(0, \infty)$  invariant, one can show that the explicit construction equals the modular construction mentioned above (cf. [Lon08b]).

**Proposition 3.2.** *Let  $(F, \langle \cdot, \cdot \rangle)$  be an Euclidean space, then there is a local Möbius covariant net of standard subspace  $H_F$  on the Hilbert space  $\mathcal{H}_{0,F}$ .*

We remark that by the geometric modular action follows that the net is *Haag dual*, i.e.  $H_F(I') = H_F(I)'$  and also the restriction to  $\mathbb{R}$  can be shown to be Haag dual, i.e.  $H_F((\mathbb{R} \setminus I)^\circ) = H_F(I)$  for  $I \in \mathbb{R}$ .

**3.2. Conformal Nets.** In this part we give the notion of a *local Möbius covariant net* of von Neumann algebras which we will simply call *conformal net*.

**Definition.** A local Möbius covariant net (conformal net)  $\mathcal{A}$  on  $S^1$  is a family  $\{\mathcal{A}(I)\}_{I \in \mathcal{I}}$  of von Neumann algebras on a Hilbert space  $\mathcal{H}$ , with the following properties:

- A. Isotony.**  $I_1 \subset I_2$  implies  $\mathcal{A}(I_1) \subset \mathcal{A}(I_2)$ .
- B. Locality.**  $I_1 \cap I_2 = \emptyset$  implies  $[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}$ .
- C. Möbius covariance.** There is a unitary representation  $U$  of Möb on  $\mathcal{H}$  such that  $U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI)$ .

**D. Positivity of energy.**  $U$  is a positive energy representation, i.e. the generator  $L_0$  (conformal Hamiltonian) of the rotation subgroup  $U(R(\theta)) = e^{i\theta L_0}$  has positive spectrum.

**E. Vacuum.** There is a (up to phase) unique rotation invariant unit vector  $\Omega \in \mathcal{H}$  which is cyclic for the von Neumann algebra  $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$ .

It holds automatically the *Reeh-Schlieder property* [FJ96], i.e.  $\Omega$  is cyclic and separating for any  $\mathcal{A}(I)$  with  $I \in \mathcal{I}$ . Further we have the *Bisognano-Wichmann property* [GF93, BGL93] saying the modular operators with respect to  $\Omega$  have geometric meaning; e.g. the modular operators for the upper circle  $I_0$  are given by the dilation  $\Delta^{it} = U(\delta(-2\pi t))$  and reflection  $J = U(r)$ , where here  $U$  is extended to  $\text{Möb}_\pm$ . For a general interval  $I \in \mathcal{I}$  the modular operators are given by a special conformal transformation  $\delta_I$  and a reflection  $r_I$  both fixing the endpoints of  $I$ . The Bisognano-Wichmann property implies *Haag duality*

$$\mathcal{A}(I)' = \mathcal{A}(I') \quad I \in \mathcal{I}$$

and it can be shown (see e.g. [GF93]) that each  $\mathcal{A}(I)$  is a type  $\text{III}_1$  factor in Connes classification [Con73]. A conformal net is *additive* [FJ96], i.e. for intervals  $I, I_1, \dots, I_n \in \mathcal{I}$  it holds

$$I \subset \bigcup_i I_i \implies \mathcal{A}(I) \subset \bigvee_i \mathcal{A}(I_i).$$

**3.2.1. Representations.** Let  $\mathcal{A}$  be a conformal net on a Hilbert space  $\mathcal{H}$ . A *covariant representation*  $\pi = \{\pi_I\}_{I \in \mathcal{I}}$  is a family of representations  $\pi_I$  of  $\mathcal{A}(I)$  on a fixed Hilbert space  $\mathcal{H}_\pi$  which fulfill:

$$\begin{aligned} \pi_I \upharpoonright_{\mathcal{A}(I_0)} &= \pi_{I_0} & I_0 \subset I \\ \text{Ad } U_\pi(g) \circ \pi_I &= \pi_{gI} \circ \text{Ad } U(g) \end{aligned}$$

where  $U_\pi$  is a unitary representation of the universal covering group  $\widetilde{\text{Möb}}$  of  $\text{Möb}$  with positive energy. We assume  $\mathcal{H}_\pi$  to be separable and this implies that  $\pi$  is *locally normal*, namely  $\pi_I$  is normal for all  $I \in \mathcal{I}$ . A representation  $\rho$  is called *localized* in some interval  $I_0 \in \mathcal{I}$  if  $\mathcal{H}_\rho = \mathcal{H}$  and  $\rho_{I_0'} = \text{id}_{\mathcal{A}(I_0')}$ . Due to the type  $\text{III}_1$  factor property, each representation  $\pi$  is localizable in any interval  $I_0 \in \mathcal{I}$ , namely there is a representation  $\rho$  which is unitary equivalent to  $\pi$  and localized in  $I_0$ . If  $\rho$  is localized in  $I_0 \in \mathcal{I}$  then by Haag duality for every  $I \in \mathcal{I}$  with  $I \supset I_0$  it is  $\rho_I(\mathcal{A}(I)) \subset \mathcal{A}(I)$ , in other words  $\rho_I$  is an endomorphism of  $\mathcal{A}(I)$ . Let  $\rho$  be a (covariant) representation localized in  $I_0$ . By a *local cocycle* [Lon03] localized in a proper interval  $I \supset I_0$ , we mean an assignment of a symmetric neighbourhood  $\mathcal{U}$  of the identity of  $\widetilde{\text{Möb}}$  such that  $I_0 \cup gI_0 \subset I$  for all  $g \in \mathcal{U}$  and a strongly continuous unitary valued map  $g \in \mathcal{U} \mapsto z_\rho(g) \in \mathcal{A}(I)$  such that with  $\alpha_g := \text{Ad } U(g)$ :

$$\begin{aligned} z_\rho(gh) &= z_\rho \alpha_g(z_\rho(h)) & g, h \in \mathcal{U} \\ \text{Ad } z_\rho(g)^* \circ \rho_{\tilde{I}}(a) &= \alpha_g \circ \rho_{g^{-1}\tilde{I}} \circ \alpha_{g^{-1}}(a) & g \in \mathcal{U}, a \in \mathcal{A}(\tilde{I}) \end{aligned}$$

for some open interval  $\tilde{I} \in \mathcal{I}$  with  $\tilde{I} \supset \overline{I}$ . By covariance and Haag duality there exist a local cocycle given by

$$z_\rho(g) = U_\rho(g)U(g)^* \in \mathcal{A}(I) \quad g \in \mathcal{U}.$$

**3.2.2. Conformal subnets.** Let  $\mathcal{A}$  be a conformal net and  $U$  its associated positive energy representation of  $\text{Möb}$ . We call a family  $\{\mathcal{B}(I)\}_{I \in \mathcal{I}}$  with  $\mathcal{B}(I) \subset \mathcal{A}(I)$  for all  $I \in \mathcal{I}$  a *conformal subnet* if  $\mathcal{B}$  is isotony, i.e.  $I, J \in \mathcal{I}$  with  $I \subset J$  implies  $\mathcal{B}(I) \subset \mathcal{B}(J)$  and covariant, i.e. it is  $U(g)\mathcal{B}(I)U(g)^* = \mathcal{B}(gI)$  for all  $I \in \mathcal{I}$  and  $g \in \text{Möb}$ . The structure of conformal subnets is studied in [Lon03].

Let  $e$  be the projection on the closure  $\mathcal{H}_{\mathcal{B}}$  of  $\bigvee_{I \in \mathcal{I}} \mathcal{B}(I)\Omega$ . Then  $\mathcal{B}$  is itself a conformal net on  $\mathcal{H}_{\mathcal{B}} := e\mathcal{H}$  with unitary representation  $U \upharpoonright_{\mathcal{H}_{\mathcal{B}}}$  also denoted by  $U$ , namely  $\Omega$  is cyclic for  $\mathcal{H}_{\mathcal{B}}$  by definition and all other properties are inherited by the ones of  $\mathcal{A}$ . By the Reeh–Schlieder property  $\Omega$  is cyclic and separating for all  $\mathcal{B}(I)$  with  $I \in \mathcal{I}$  and in particular  $e$  is the Jones projection (see e.g. [LR95]) of the inclusion  $\mathcal{B}(I) \subset \mathcal{A}(I)$ .

**Lemma 3.3.** *Let  $\mathcal{B}$  be a conformal subnet of  $\mathcal{A}$ . If  $e = 1$  then the conformal nets  $\mathcal{B}$  and  $\mathcal{A}$  are identic.*

*Proof.* Let  $I \in \mathcal{I}$ . Then it is  $\mathcal{B}(I) \subset \mathcal{A}(I)$  and by the Bisognano–Wichmann property the modular group of  $\mathcal{A}(I)$  with respect to the vector state of  $\Omega$  is given by  $\sigma_t = \text{Ad } U(\delta_I(-2\pi t))$  and by covariance of  $\mathcal{B}$  it leaves  $\mathcal{B}(I)$  invariant. By Takesaki's Theorem [Tak03, Theorem IX.4.2.] there exists a normal conditional expectation from  $\mathcal{A}(I)$  onto  $\mathcal{B}(I)$  which has to be the identity of  $\mathcal{A}$  by  $e = 1$ .  $\square$

**3.2.3. Completely rational conformal nets.** A conformal net  $\mathcal{A}$  is said to be *strongly additive* if for  $I_1, I_2 \in \mathcal{I}$  adjacent intervals and  $I = (I_1 \cup I_2)'' = \overline{I_1} \cup \overline{I_2}^\circ \in \mathcal{I}$  it holds

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I).$$

The net  $\mathcal{A}$  is called *split* if for  $I_0, I \in \mathcal{I}$  with  $\overline{I_0} \subset I$  the inclusion  $\mathcal{A}(I_0) \subset \mathcal{A}(I)$  is a split inclusion, namely there exist an intermediate type I factor  $M$  such that  $\mathcal{A}(I_0) \subset M \subset \mathcal{A}(I)$  or equivalently  $\mathcal{A}(I_0) \vee \mathcal{A}(I)'$  is canonically isomorphic to  $\mathcal{A}(I_0) \otimes \mathcal{A}(I)'$ . Let  $I_1, I_3 \in \mathcal{I}$  be two intervals with disjoint closure and  $I_2, I_4 \in \mathcal{I}$  the two components of  $(I_1 \cup I_3)'$ , in other words the intervals  $I_1, \dots, I_4$  divide the circle into four parts. Then we denote by  $\mu_{\mathcal{A}}$  the Jones–Kosaki index [Kos98] of the inclusion

$$\mathcal{A}(I_1) \vee \mathcal{A}(I_3) \subset (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))' \quad (1)$$

which does not depend on the intervals  $I_i$ . Finally the net  $\mathcal{A}$  is called *completely rational* if it is strongly additive, split and  $\mu_{\mathcal{A}} < \infty$ . In [KLM01] it is shown that the index of the inclusion (1) is the global index associated with all sectors and the category of representations form a modular tensor category, where each sector is a direct sum of sectors with finite dimension.

**3.3. Second Quantization Nets.** By second quantization of a net of standard subspaces we become a net of von Neumann algebras.

Let  $\mathcal{H}$  be a Hilbert space and  $\omega(\cdot, \cdot) = \text{Im}(\cdot, \cdot)$  the sesquilinear form. There are unitaries  $W(f)$  for  $f \in \mathcal{H}$  fulfilling

$$W(f)W(g) = e^{-i\omega(f,g)}W(f+g) = e^{-2i\omega(f,g)}W(g)W(f).$$

and acting naturally on the Bosonic Fock space  $e^{\mathcal{H}}$  over  $\mathcal{H}$ . This space is given by  $e^{\mathcal{H}} = \sum_{n=0}^{\infty} P_n \mathcal{H}^{\otimes n}$ , where  $P_n$  is the projection  $P_n(x_1 \otimes \dots \otimes x_n) = 1/n! \sum_{\sigma} x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(n)}$  where the sum goes over all permutation. The set of coherent vectors  $e^h := \bigoplus_{n=0}^{\infty} h^{\otimes n} / \sqrt{n!}$  with  $h \in \mathcal{H}$  is total in  $e^{\mathcal{H}}$  and it is  $(e^f, e^h) = e^{(f,h)}$ . The vacuum is given by  $\Omega = e^0$  and the action of  $W(f)$  is given by  $W(f)e^0 = e^{-\frac{1}{2}\|f\|^2} e^f$ , in other words the vacuum representation  $\phi(\cdot) = (\Omega, \cdot \Omega)_{\mathcal{H}_F}$  is characterized by  $\phi(W(f)) = e^{-\frac{1}{2}\|f\|^2}$ .

For a real subspace  $H \subset \mathcal{H}_{0,F}$  we define the von Neumann algebra

$$R(H) = \{W(f) : f \in H\}'' \subset B(e^{\mathcal{H}}).$$

The map  $R$  has the following properties

**Proposition 3.4** ([Lon08a]).

(1) *Let  $H, K \subset \mathcal{H}$  be real linear subspaces. Then  $R(K) = R(H)$  iff  $\bar{K} = \bar{H}$ .*



- (2) Let  $H$  be closed.  $H$  is separating or cyclic iff  $R(H)$  is separating or cyclic, respectively.
- (3) Let  $H$  be standard, then the modular unitaries  $\Delta_{R(H)}^{\text{it}}$  and the modular conjugation  $J_{R(H)}$  associated with  $(R(H), \Omega)$  are given by

$$\Delta_{R(H)}^{\text{it}} = \Gamma(\Delta_H^{\text{it}}), \quad J_{R(H)} = \Gamma(J_H)$$

and in particular  $R(H') = R(H)'$ .

Let  $U$  be a unitary in  $B(\mathcal{H})$  then  $\Gamma(U) = \bigoplus_{n=0}^{\infty} U^{\otimes n}$  acts on coherent states by  $\Gamma(U)e^h = e^{Uh}$  and is therefore a unitary (cf. [Gui11]) on  $e^{\mathcal{H}}$ . These second quantization unitaries implement Bogoliubov automorphism, namely  $\Gamma(U)W(f)\Gamma(U)^* = W(Uf)$ .

**Proposition 3.5** (Second quantization nets [Lon08a]). *Let  $\{H(I)\}_{I \in \mathcal{I}}$  be a local Möbius covariant net of standard subspace on  $\mathcal{H}$ . Then  $\mathcal{A}(I) = R(H(I))$  defines a local Möbius covariant net (of von Neumann algebras) on  $e^{\mathcal{H}}$ .*

Let  $(F, \langle \cdot, \cdot \rangle)$  be an Euclidean space and  $\mathcal{I} \ni I \mapsto H_F(I) \subset \mathcal{H}_{0,F}$  the net of standard subspaces from Proposition 3.2. Then we denote by  $\mathcal{A}_F$  the local Möbius covariant net on  $\mathcal{H}_F := e^{\mathcal{H}_{0,F}}$  called the *Abelian current net over  $F$*  given by  $\mathcal{A}_F(I) := R(H_F(I))$ . If  $U_0$  is the action of Möb on  $\mathcal{H}_{0,F}$  then the action on  $\mathcal{H}_F$  is given by  $U(g) := \Gamma(U_0(g))$ . In the case  $F = \mathbb{R}$  the net is also called the *U(1)-current net* and was treated in an operator algebraic setting first in [BMT88]. We remark that  $\mathcal{A}_F$  is clearly equivalent to the  $n$ -fold tensor product of the U(1)-current net.

**3.3.1. Representations.** Let  $\ell \in C^\infty(S_1, F)$  with support in some  $I_0 \in \mathcal{I}$ . Then we define for  $I \in \mathcal{I}$  with  $I_0 \subset I$

$$\rho_{\ell, I}(W(f)) = e^{-i \int \langle f, \ell \rangle} W(f)$$

where we have chosen a representant  $f$  of  $[f] \in \mathcal{H}_{F,0}$  with  $f|_{I'} \equiv 0$ . This defines a representation localized in  $I_0$ . This representation is covariant with local cocycle localized in  $I \supset I_0$  and  $\mathcal{U}$  a symmetric neighbourhood of the identity of Möb such that  $I_0 \cup gI_0 \subset I$  for all  $g \in \mathcal{U}$  given by  $z(g) = W(L - L_g)$  where  $L$  is a primitive of  $\ell$ , i.e.  $L'(\theta) = \ell(\theta)$  and  $L_g(\theta) = g_*L(\theta) = L(g^{-1}\theta)$ .

Two representations are equivalent if they have the same charge, which is given for  $\rho_\ell$  by

$$q_\ell = \int_0^{2\pi} \ell(\theta) \frac{d\theta}{2\pi} = \int \ell \in F,$$

namely for  $q_\ell = q_m$  it is  $z\rho_\ell = \rho_m z$  with unitary intertwiner  $z = W(M - L)$  where  $M - L \in \mathcal{H}_{F,0}$  is a primitive of  $m - \ell$ . In other words the sectors depend only on this charge  $q \in F$  and we denote the sector by  $[q]$  with obvious fusion rules  $[q] \times [r] = [q + r]$ . We note that because there are infinitely many sectors (with dimension 1) the index of the inclusion (1) is infinite and the nets cannot be completely rational.

Equivalently the conformal net can be regarded as coming from a projective positive energy representation of the group  $\mathbb{L}F$ .

**3.3.2. Abelian currents from central extensions.** Basically to fix notation, we recall some facts about projective representations. If  $\pi$  is a projective representation of a group  $G$  on a Hilbert space  $\mathcal{H}$ , then there is a 2-cocycle with  $c : G \times G \rightarrow \mathbb{T} \subset \mathbb{C}$  given by

$$\pi(g)\pi(h) = c(g, h)\pi(gh) \quad \text{for all } g, h \in G$$

fulfilling the cocycle relation  $c(h, k)c(g, hk) = c(g, h)c(gh, k)$ , which follows from associativity. Two representations are equivalent if and only if there is a coboundary

$$b_f(g, h) = \frac{f(g)f(h)}{f(gh)} \quad (2)$$

where  $f : G \rightarrow \mathbb{T}$  such that

$$c_2(g, h) = b_f(g, h)c_1(g, h) \iff \pi_2(g) = f(g)\pi_1(g).$$

If  $G \cong \mathbb{Z}^n$  this is true if and only if  $\hat{c}_1 = \hat{c}_2$  (see for example [Kac98, Lemma 5.5] cf. also [DHR69, Lemma A.1.2]) where  $\hat{c}(g, h) = c(g, h)c(h, g)^{-1}$  is the *commutator map* or *antisymmetric part* of a cocycle  $c$ . The following Lemma will be useful showing the equivalence of two cocycles.

**Lemma 3.6.** *Let  $G = G_1 \times G_2$  be an Abelian group and  $c, c' \in Z^2(G, \mathbb{T})$  be two 2-cocycle and  $c_i, c'_i \in Z^2(G_i, \mathbb{T})$  their restrictions to  $G_i \times G_i$  for  $i = 1, 2$ . If  $[c_i] = [c'_i] \in H^2(G_i, \mathbb{T})$  then  $\hat{c} = \hat{c}'$  implies  $[c'] = [c] \in H^2(G, \mathbb{T})$ .*

*Proof.* The proof is basically [DHR69, Proof of Lemma A.1.2.]. Because  $c \mapsto \hat{c}$  is an homomorphism it is enough to show that for  $c \in Z^2(G, \mathbb{T})$ : if  $\hat{c} = 1$  and  $c_i(g_i, h_i) = b_i(g_i)b_i(h_i)/b_i(g_i h_i)$  then  $c \in B^2(G, \mathbb{T})$ , i.e.  $c = \delta b$  with  $b \in Z^1(G, \mathbb{T})$ . Indeed, setting

$$b(g_1 g_2) = \frac{b_1(g_1)b_2(g_2)}{c(g_1, g_2)} \equiv \frac{b_1(g_1)b_2(g_2)}{c(g_2, g_1)}$$

for  $g_i \in G_i$  we calculate using the cocycle relation:

$$\begin{aligned} c(g_1 g_2, h_1 h_2) &= \frac{c(g_1, g_2 h_1 h_2)c(g_2, h_1 h_2)}{c(g_1, g_2)} \\ &= \frac{c_1(g_1, h_1)c(g_1 h_1, g_2 h_2)c(g_2 h_2, h_1)c_2(g_2, h_2)}{c(g_1, g_2)c(h_1, g_2 h_2)c(h_2, h_1)} \\ &= \frac{b(g_1 g_2)b(h_1 h_2)}{b(g_1 h_1 g_2 h_2)}. \end{aligned}$$

□

Equivalently to say that  $\pi$  is a projective representation there is a true representation also denoted by  $\pi$  of the group  $\tilde{G} = G \times \mathbb{T}$  with multiplicative law  $(g_1, t_1)(g_2, t_2) = (g_1 g_2, c(g_1, g_2)t_1 t_2)$  given by  $\pi(g, t) = t\pi(g)$ . One calls  $\tilde{G}$  a *central extension* of  $G$ .

Let  $G$  be a Lie group and  $\pi$  a continuous projective unitary representation of  $LG$  on a Hilbert space  $\mathcal{H}$ . We assume that there is an action of the rotation, i.e.  $\mathbb{T}$  acts unitarily on  $\mathcal{H}$  by  $U$  such that  $U(\theta)\pi(f)U(\theta)^* = \pi(R_\theta f)$  where  $R_\theta f(\theta') = f(\theta' - \theta)$  for  $f \in LG$ . In other words we assume  $\pi$  extends to a representation of  $LG \rtimes \mathbb{T}$ . Then  $\pi$  is called *positive energy* (cf. [Seg81, PS86]) if

$$\mathcal{H} = \oplus_{n \geq 0} \mathcal{H}_n, \quad \mathcal{H}_n = \{x \in \mathcal{H} : U(\theta)x = e^{in\theta}\} \quad (3)$$

with  $\dim \mathcal{H}_n < \infty$  and<sup>2</sup>  $\mathcal{H}_0 \neq \{0\}$ . That means the generator  $L_0$  of  $U(\theta) = e^{i\theta L_0}$  has positive spectrum.

Let  $\mathcal{L}F$  be the central extension of  $LF$  defined by the cocycle

$$c_F(f, g) = e^{-i\omega(f, g)} = e^{-i/2 \int \langle f', g \rangle}.$$

Then the conformal net  $\mathcal{A}_F$  constructed above can be regarded as the conformal net associated with a positive energy representation of  $LF$  with cocycle  $c_F$  or equivalently a (true) positive energy representation of  $\mathcal{L}F$ . For  $I \in \mathcal{I}$  we denote by  $\mathbf{L}_I F$  all loops with support in  $I$ .

<sup>2</sup> this can be obtained by multiplying a given representation of  $\mathbb{T}$  with a character of  $\mathbb{T}$

**Proposition 3.7.** *Let  $\mathcal{H}_{0,F} = \overline{\mathbb{L}F}/F$  be the one-particle space associated with  $(F, \langle \cdot, \cdot \rangle)$ . There is unitary positive energy representation  $\pi$  of  $\mathcal{L}F$  on the Fock space  $\mathcal{H}_F \equiv e^{\mathcal{H}_{0,F}}$  given by:  $\pi_0 : \mathcal{L}F \equiv \mathbb{L}F \times \mathbb{T} \ni (f, c) \mapsto c \cdot W([f])$ , where  $[f] \in \mathbb{L}F/F \subset \mathcal{H}_F$ . In particular it is  $\mathcal{A}_F(I) = \pi_0(\mathbb{L}_I F)''$ .*

*Proof.*  $W([f])$  is unitary by construction. Obviously by the Weyl commutation relations  $\pi$  is a representation of  $\mathcal{L}F$  with the given cocycle. Let  $L_0$  be the positive generator of the rotations on  $\mathcal{H}_{0,F}$ . The generator of the rotation on  $\mathcal{H}_F$  is then given by  $\tilde{L}_0 = 1 \oplus L_0 \oplus (L_0 \otimes 1 + 1 \otimes L_0) \oplus \dots$  and in particular positive.

Further for  $I \in \mathcal{I}$  by construction  $\pi(\mathbb{L}_I F)'' = R(\mathbb{L}_I F/F)$  which equals  $\mathcal{A}(I) \equiv R(H_F(I))$  by Proposition 3.4 and because  $\mathbb{L}_I F$  is dense in  $H_F(I)$  again by construction.  $\square$

**3.4. Conformal Nets Associated with Lattices.** We want to consider local extensions of the net  $\mathcal{A}_F$  associated with Abelian currents with values in  $F$ . The case of  $F = \mathbb{R}$  (one current) was treated in [BMT88] and the extensions are given by a charge  $g = \sqrt{2N}$  with  $N \in \mathbb{N}$ . The general case was elaborated in [Sta95] with the result that the extension are given by even integral lattice (for  $n = 1$  the lattice is  $g\mathbb{Z}$ ). The same lattice models were also examined in [DX06], where they are equivalently defined as a positive energy representation of the loop group of the torus associated with the lattice. This gives a connection to the representations of loop groups at level 1 [Seg81, PS86] for simply laced Lie groups. The lattice models are well known in the framework of vertex operator algebras. For a treatment of lattice models in vertex operator algebras and its connection to Kac–Moody algebras we refer e.g. to [Kac98, Chapter 5.4].

Let  $L$  be an *integral (positive) lattice*, i.e. a free  $\mathbb{Z}$ -module with positive-definite integral bilinear form  $\langle \cdot, \cdot \rangle : L \times L \rightarrow \mathbb{Z}$ . A lattice is called *even* if  $\langle \alpha, \alpha \rangle \in 2\mathbb{N}$  for all  $\alpha \in L$ , and we note that an even lattice is necessarily integral. To a lattice  $L$  we relate an Euclidean space  $(F, \langle \cdot, \cdot \rangle)$  where  $F = L \otimes_{\mathbb{Z}} \mathbb{R}$  and the scalar product  $\langle \cdot, \cdot \rangle$  is continued to  $F \times F \rightarrow \mathbb{R}$  by linearity. The dimension  $n = \dim F$  is called the *rank* (assumed to be finite).

Equivalently we can view an even lattice  $L$  as a free discrete subgroup of a finite dimensional Euclidean space  $(F, \langle \cdot, \cdot \rangle)$  which spans  $F$  and satisfies  $\langle \alpha, \alpha \rangle \in 2\mathbb{N}$  for all  $\alpha \in L$ . Let  $L$  be even and  $L^* := \{x \in V : \langle x, L \rangle \subset \mathbb{Z}\}$  be the dual lattice [CS98]. It is not necessarily an integer lattice and can canonically be identified with  $\text{Hom}(L, \mathbb{Z})$  by the scalar product. It is  $L \subset L^*$  and it can be shown that the group  $L^*/L$  is finite. In the case  $L^* = L$  the lattice is called *self-dual* or *unimodular* and in this can be the case only for rank  $n \in 8\mathbb{N}$ .

With an even lattice  $L$  we associate a torus  $T = F/2\pi L$ , and we will represent elements by  $e^{if}$  with  $f \in F$  and  $e^{if} = 1$  if and only if  $f \in L$ , formally

$$F/2\pi L \hookrightarrow T, [t] \mapsto e^{it}.$$

**3.4.1. Loop group associated with a torus.** Let  $\mathbb{L}T = C^\infty(\mathbb{S}^1, T)$  the loop group associated with the torus  $T$ . We write  $e^{if}$  for an element in  $\mathbb{L}T$  where we mean the function  $e^{i\theta} \mapsto e^{if(\theta)}$  and  $f : \mathbb{R} \rightarrow F$  is a smooth function such that the winding number

$$\Delta_f := \frac{1}{2\pi}(f(\theta + 2\pi) - f(\theta))$$

is constant and takes values in  $L$ . In particular  $f_\circ : \theta \mapsto f(\theta) - \Delta_f \cdot \theta$  is a periodic function and we can decompose

$$f(\theta) = \Delta_f \cdot \theta + f_0 + \sum_{n \in \mathbb{Z}^*} f_n e^{in\theta}$$

where we call  $f_0$  the *zeroth-mode*.

We are interested in projective positive energy representations of  $\mathcal{LT}$  or equivalently representations of a central extension:

$$1 \longrightarrow \mathbb{T} \longrightarrow \mathcal{LT} \longrightarrow \mathcal{LT} \longrightarrow 1$$

which are given by a cocycle  $c : \mathcal{LT} \times \mathcal{LT} \longrightarrow \mathbb{T}$  specified in the following.

It is well-known (see for example [Kac98]) that there exist a bilinear form  $b : L \times L \longrightarrow \mathbb{Z}_2$  such that

$$b(\alpha, \alpha) = \frac{1}{2} \langle \alpha, \alpha \rangle \quad \text{for all } \alpha \in L,$$

e.g. if  $\{\alpha_1, \dots, \alpha_n\}$  is a basis of  $L$  one can choose

$$b(\alpha_i, \alpha_j) = \begin{cases} \langle \alpha_i, \alpha_j \rangle \mod 2 & i < j \\ \frac{1}{2} \langle \alpha_i, \alpha_i \rangle \mod 2 & i = j \\ 0 & i > j. \end{cases}$$

Therefor it exists a bimultiplicative map  $\varepsilon(\alpha, \beta) : L \times L \longrightarrow \{+1, -1\} \cong \mathbb{Z}_2$ , satisfying  $\varepsilon(\alpha, \alpha) = (-1)^{\langle \alpha, \alpha \rangle / 2}$ . Such a map is a 2-cocycle satisfying:

$$\begin{aligned} \varepsilon(\alpha, \beta + \gamma) \varepsilon(\beta, \gamma) &= \varepsilon(\alpha, \beta) \varepsilon(\alpha + \beta, \gamma) \\ \varepsilon(\alpha, \beta) \varepsilon(\beta, \alpha) &= (-1)^{\langle \alpha, \beta \rangle}. \end{aligned}$$

Now we specify the central extension  $\mathcal{LT}$  by choosing a 2-cocycle  $c : \mathcal{LT} \times \mathcal{LT} \longrightarrow \mathbb{T}$  as in [Seg81]

$$\begin{aligned} c(e^{if}, e^{ig}) &\equiv c(f, g) = \varepsilon(\Delta_f, \Delta_g) e^{-iS(f, g)} \\ 2 \cdot S(f, g) &= \int_0^{2\pi} \langle f'(\theta), g(\theta) \rangle \frac{d\theta}{2\pi} + \langle \Delta_f, g(0) \rangle. \end{aligned} \tag{4}$$

We note that the central extension (up to equivalence) does not depend on the explicit choice of the 2-cocycle in its equivalence class. Further we write the relations in  $\mathcal{LT}$  formally as  $e^{if} e^{ig} = c(e^{if}, e^{ig}) e^{i(f+g)}$ . It is straightforward to verify the following relations.

**Lemma 3.8** (cf. [DX06]). *Let  $e^{if}, e^{ig} \in \mathcal{LT}$ , then we have the following relations in  $\mathcal{LT}$ .*

$$e^{if} e^{ig} (e^{if})^{-1} = e^{i\pi \langle \Delta_f, \Delta_g \rangle} e^{-i \int \langle f'_1, g_1 \rangle} e^{-i \langle \Delta_f, g_0 \rangle + i \langle \Delta_g, f_0 \rangle} e^{ig}$$

*Proof.* We observe that  $(e^{if})^{-1} = c(e^{if}, e^{-if})^{-1} e^{-if} = c(f, f) e^{-if}$  and we get:

$$\begin{aligned} e^{if} e^{ig} (e^{if})^{-1} &= c(f, g) c(g, -f) c(f, -f) c(f, f) e^{ig} \\ &= c(f, g) c(g, f)^{-1} e^{ig} \\ &= (-1)^{\langle \Delta_f, \Delta_g \rangle} e^{-i(S(f, g) - S(g, f))} e^{ig}. \end{aligned} \tag{5}$$

Using  $f(\theta) = \Delta_f \cdot \theta + f_0 + f_1(\theta)$  and  $g(\theta) = \Delta_g \cdot \theta + g_0 + g_1(\theta)$  we have

$$\begin{aligned} S(f, g) &= \frac{1}{4\pi} \int_0^{2\pi} \langle f'_1(\theta), g_1(\theta) \rangle d\theta + \frac{\pi}{2} \langle \Delta_f, \Delta_g \rangle + \frac{1}{2} \langle \Delta_f, g_0 \rangle + \\ &\quad + \frac{1}{2} \langle f_1(2\pi), \Delta_g \rangle + \frac{1}{2} \langle \Delta_f, g_1(0) \rangle \end{aligned}$$

and this gives

$$\begin{aligned} S(f, g) - S(g, f) &= \frac{1}{2\pi} \int_0^{2\pi} \langle f'_1(\theta), g(\theta) \rangle d\theta + \frac{1}{2} \langle \Delta_f, g_0 \rangle - \frac{1}{2} \langle \Delta_g, f_0 \rangle \\ &\quad + \frac{1}{2} \langle \Delta_f, g(0) - g_1(0) \rangle - \frac{1}{2} \langle \Delta_g, f(0) - f_1(0) \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \langle f'_1(\theta), g_1(\theta) \rangle d\theta + \langle \Delta_f, g_0 \rangle - \langle \Delta_g, f_0 \rangle \end{aligned}$$

which inserted in (5) completes the proof.  $\square$

The following Proposition is proved in [DX06, Proposition 3.4] and shows that the central extension is local, i.e. loops supported in disjoint intervals commute.

**Proposition 3.9** (Locality cf. [DX06, Prop. 3.4]). *If  $\text{supp } e^{if} \cap \text{supp } e^{ig} = \emptyset$  then  $e^{if} e^{ig} = e^{ig} e^{if}$ .*

For  $I \in \mathcal{I}$  we denote by  $\mathcal{L}_I T = \{e^{if} \in \mathcal{L}T : \text{supp } e^{if} \subset I\}$  all loops with support in  $I$  and by  $\mathcal{L}_I T$  the preimage of  $\mathcal{L}_I T$  under the covering map. Using this locality and well-known results of positive energy representations of loop groups it is shown in [DX06] that there is a conformal net associated with  $\mathcal{L}T$ , precisely:

**Proposition 3.10** (Local conformal net associated with  $\mathcal{L}T$  cf. [DX06]). *There is a correspondence between the elements of  $L^*/L$  and positive energy representation of  $\mathcal{L}T$ . Let  $\pi_{(L,0)}$  be the (vacuum) representation corresponding to  $[0] \in L^*/L$ , then*

$$I \longmapsto \mathcal{A}_{\mathcal{L}T}(I) := \pi_{(L,0)}(\mathcal{L}_I T)''$$

*is a completely rational conformal net with  $\mu$ -index  $\mu = |L^*/L|$  and has  $\mu$  sectors of statistical dimension 1 corresponding to the positive energy representations of  $\mathcal{L}T$ .*

*Proof.* For the first statement see [DX06, Lemma 3.5] and [PS86, Section 9.5].  $\mathcal{A}_{\mathcal{L}T}$  is a local net by Proposition 3.9 and [DX06, Proposition 3.1] shows that it is a strongly additive conformal net fulfilling the split property. [DX06, Proposition 3.15] shows the correspondence between sectors and elements of  $L/L^*$  and the  $\mu$  index is given in [DX06, Corollary 3.19].  $\square$

**Remark 3.11.** *We note that the construction depend only on  $L$  and we denote this net also by  $\mathcal{A}_L$ , the conformal net associated with the lattice  $L$ .*

The rest of the section we give the construction in a more explicit manner. In particular the Hilbert space  $\mathcal{H}_L$  of  $\mathcal{A}_L$  can naturally be identified with  $L$  copies of the Hilbert space  $\mathcal{H}_F$  of  $\mathcal{A}_F$  (more precisely  $\ell^2(L, \mathcal{H}_F)$ ) where  $F = L \otimes_{\mathbb{Z}} \mathbb{R}$ . This enables us to show that  $\mathcal{A}_L(I)$  is a crossed product of  $\mathcal{A}_F(I)$  with  $L$ .

The identity component  $(\mathcal{L}T)_0$  of  $\mathcal{L}T$  can be identified with  $H_F \times T$ , where  $H_F = \mathbb{L}F/F \times \mathbb{T}$  is the Heisenberg group with multiplication law  $(f, c_1)(g, c_2) = (f + g, e^{-i/2\omega(f,g)} c_1 c_2)$ . The representation  $\pi_0$  of  $\mathcal{L}F$  is a representation of  $H_F$  because the constant loops lie in the kernel of the representation and it turns out to be the unique irreducible representation (cf. proof of 9.5.10 [PS86]) with positive energy. Let  $\tilde{W} = H_F \times F$  (the idea is to add an operator  $Q$  which measures the charge). All irreducible representations of positive energy of  $\tilde{W}$  are classified by a charge  $\alpha \in F$  and are of the form  $(\pi_\alpha, (\mathcal{H}_F)_\alpha)$  given by  $\pi_\alpha(f, v) = e^{-i2\pi\langle \alpha, v \rangle} \pi_0(f)$ . As a set it is  $\mathcal{L}T \equiv (\mathcal{L}T)_0 \times L$  and it is shown in [PS86] that all irreducible representations of  $\mathcal{L}T$  of positive energy are given by points  $\lambda \in L^*/L$  and are acting on the Hilbert space

$$\mathcal{H}_{(L,\lambda)} = \bigoplus_{\alpha \in \lambda + L} (\mathcal{H}_F)_\alpha.$$

The Hilbert space  $\mathcal{H}_{(L,0)}$  on which  $\mathcal{LT}$  acts is graded by the lattice  $L$ , and we call  $\alpha$  the charge of the subspace  $(\mathcal{H}_F)_\alpha$  of  $\mathcal{H}_{(L,0)}$ . We define for  $\alpha \in L$  charge shift operators  $\Gamma_\alpha$  by  $(\Gamma_\alpha x)_\beta = x_{\beta-\alpha}$  and introduce the unbounded charge operator  $Q$  satisfying  $(Qx)_\alpha = \alpha x$ . The  $\Gamma_\alpha$  does not fulfill exactly the commutation relations suitable for the representation of  $\mathcal{LT}$ . But the commutation relations between different  $\Gamma_\alpha$  can be changed by a so called Klein transformation. Let  $\eta : L \times L \rightarrow \mathbb{T}$  be a bimultiplicative map (2-cocycle) and  $\tilde{\Gamma}_\alpha = \eta(-Q, \alpha)\Gamma_\alpha$  then we have:

**Lemma 3.12.**  $\alpha \mapsto \tilde{\Gamma}_\alpha$  defines a representation of the central extension  $\tilde{L}$  of  $L$  by the cocycle  $\eta(\cdot, \cdot)$ .

*Proof.* We note that  $\tilde{L} = L \times \mathbb{T}$  with multiplication law

$$(\alpha, c)(\beta, d) = (\alpha + \beta, \eta(\alpha, \beta)cd)$$

and the representation is obtained by applying  $(\alpha, c) \mapsto c\Gamma_\alpha$ . Indeed we calculate

$$\begin{aligned} \tilde{\Gamma}_\alpha \tilde{\Gamma}_\beta &= \eta(-Q, \alpha)\Gamma_\alpha \eta(-Q, \beta)\Gamma_\beta \\ &= \eta(-Q, \alpha)\eta(-Q + \alpha, \beta)\Gamma_\alpha \Gamma_\beta \\ &= \eta(\alpha, \beta)\eta(-Q, \alpha + \beta)\Gamma_{\alpha+\beta} \\ &= \eta(\alpha, \beta)\tilde{\Gamma}_{\alpha+\beta}. \end{aligned}$$

□

We choose  $\eta(\alpha, \beta) = c(e^{it_\alpha}, e^{it_\beta})$  where  $t_\alpha(\theta) = \alpha \cdot \theta$  and get a representation of  $\{(e^{it_\alpha}, c) : \alpha \in L\} \subset \mathcal{LT}$  by  $(e^{it_\alpha}, c) \mapsto c \cdot \tilde{\Gamma}_\alpha$ .

**Proposition 3.13.** The vacuum representation of  $\mathcal{LT}$  acts by the above construction irreducible on

$$\mathcal{H}_L := \mathcal{H}_{(L,0)} \equiv \bigoplus_{\alpha \in L} (\mathcal{H}_F)_\alpha$$

i.e. the local net  $\mathcal{A}_L$  acts on  $\mathcal{H}_L$ .

*Proof.* Let  $e^{if}, e^{ig} \in \mathcal{LT}$ . We note first that for  $f = f_\Delta + f_0 + f_1$  with  $f_\Delta(\theta) = \Delta_f \cdot \theta$  and  $f_0$  zero-mode like before, we have  $e^{if} = k e^{if_0} e^{if_1} e^{if_\Delta}$  with an irrelevant phase  $k = e^{i/2 \langle f_1(2\pi), \Delta_f \rangle} \in \mathbb{T}$ .

We claim that

$$\pi'(e^{if}) = e^{i \langle f_0, Q \rangle} W(f_1) \tilde{\Gamma}_{\Delta_f}$$

defines a projective representation of  $\mathcal{LT}$  with a to  $c(\cdot, \cdot)$  equivalent cocycle  $c'(\cdot, \cdot)$ . Then there exists a coboundary  $b_h(\cdot, \cdot)$  like in (2) with  $c(f, g) = b_h(f, g)c'(f, g)$  and  $\pi(f) = h(f)\pi'(f)$  is the wanted representation.

We can write  $\mathcal{LT} = \mathcal{LT}_0 \times L$  where  $\mathcal{LT}_0$  is the connected component of the identity and  $\alpha \in L$  is identified with the loop  $t_\alpha(\theta) = \alpha \cdot \theta$ . The cocycle restricted to  $L$  are equivalent by Lemma 3.12. Further the Weyl relations give exactly the relations of the cocycle  $c(\cdot, \cdot)$ , namely

$$\begin{aligned} \pi(e^{i(f_0+f_1)})\pi(e^{i(g_0+g_1)}) &= e^{i \langle d_0, Q \rangle} W(f_1) e^{i \langle g_0, Q \rangle} W(g_1) \\ &= e^{i \langle f_0+g_0, Q \rangle} e^{-i/2 \int \langle f'_1, g_1 \rangle} W(f_1 + g_1) \\ &= e^{-i/2 \int \langle f'_1, g_1 \rangle} \pi(e^{i(f_0+g_0+f_1+g_1)}) \\ &= c(f_0 + f_1, g_0 + g_1) \pi(e^{i(f_0+g_0+f_1+g_1)}), \end{aligned}$$

so the cocycles restricted to  $\mathcal{LT}_0$  are also equal. By Lemma 3.6 it is sufficient to check that the pairwise commutation relations of  $\pi'(e^{if})$  and  $\pi'(e^{ig})$  equal the one

of  $\mathcal{LT}$  given in Lemma 3.8, indeed

$$\begin{aligned}\pi(e^{if\Delta})\pi(e^{i(g_0+g_1)})\pi(e^{if\Delta})^* &= \tilde{\Gamma}_{\Delta_f} e^{i\langle g_0, Q \rangle} W(g_1) \tilde{\Gamma}_{\Delta_f}^* \\ &= e^{i\langle g_0, Q - \Delta_f \rangle} W(g_1) \\ &= e^{-i\langle \Delta_f, g_0 \rangle} \pi(e^{i(g_0+g_1)}). \quad \square\end{aligned}$$

**Proposition 3.14.** *The local algebras  $\mathcal{A}_L(I)$  are given by a crossed product of  $\mathcal{A}_F(I)$  with  $L$ .*

*Proof.* Let  $I$  be a proper interval and  $y \in \mathbb{S}^1 \setminus \bar{I}$ . The local loop group  $\mathcal{L}_I T$  is generated by loops  $e^{if}$  with  $f(x) \in 2\pi L$  for  $x \notin I$ . We note that  $\mathcal{L}_I T = (\mathcal{L}_I T)_0 \times L$  as a set, where  $(\mathcal{L}_I T)_0$  is the connected component of the identity consisting of loops  $e^{if}$  with  $\Delta_f = 0$  and  $L$  is identified with  $\{e^{if_\alpha} : \alpha \in L\}$ , where  $t_\alpha$  are functions like above with  $\Delta_{t_\alpha} = \alpha$ . We choose a basis  $\{\alpha_i\}$  of  $L$  and some smooth “step function”  $M : \mathbb{R} \rightarrow \mathbb{R}$  with  $M(\theta + 2\pi) = M(\theta)$  and with  $\Delta_M = 1$ , such that for  $x \notin I$  it is  $M(x) \in \mathbb{Z}$  and therefore  $m(x) := M'(x) = 0$ . The loop  $e^{iM\alpha_i}$  has winding number  $\alpha_i$  and implements an automorphism  $\beta_i$  of  $\pi((\mathcal{LT})_0)''$

$$\begin{aligned}\pi(e^{if}) &= e^{i\langle Q, f_0 \rangle} W([f_1]) =: \tilde{W}(f) \\ \beta_i &:= \text{Ad } \pi(e^{iM\alpha_i}) \\ \beta_i(\tilde{W}(f)) &= e^{-i \int \langle f, m\alpha_i \rangle} \tilde{W}(f).\end{aligned}$$

which defines an automorphic action  $\beta$  of  $L$  on the algebra  $\pi((\mathcal{LT})_0)''$ . We note that with the notation from above  $\mathcal{H}_F \cong (\mathcal{H}_F)_0 = \overline{\pi((\mathcal{LT})_0)\Omega} \subset \mathcal{H}_{(L,0)}$  and denote by  $\pi_F : (\mathcal{LT})_0 \rightarrow \text{U}((\mathcal{H}_F)_0)$  the representation of  $(\mathcal{LT})_0$  on  $(\mathcal{H}_F)_0$  obtained by restriction of  $\pi$ . By construction we get  $\mathcal{A}_F(I) = \pi_F((\mathcal{L}_I T)_0)''$ . This is the vacuum representation and it is  $W([f]) = \tilde{W}(f)$ . Finally we can see  $\beta_i$  as an automorphism of  $\mathcal{A}_F(I)$

$$\beta_i(W([f])) = e^{-i \int \langle f - f(y), m\alpha_i \rangle} W([f])$$

and it is clear that  $\pi(\mathcal{L}_I T)'' = \mathcal{A}_F(I) \rtimes_\beta L$ , where the action is free and faithful.  $\square$

**Remark 3.15.** *By construction we have that  $\mathcal{A}_{L \oplus Q} \cong \mathcal{A}_L \otimes \mathcal{A}_Q$ .*

The adjoint action of a (localized) loop  $e^{if}$  with  $\Delta_f = \lambda \in L^*$  gives a localized endomorphism of  $\mathcal{A}_L$  which belongs to the sector  $[\lambda] \in L^*/L$ . The conformal spin is well known to be  $e^{i\pi\langle \lambda, \lambda \rangle}$ .

**Proposition 3.16.** *Let  $L \subset Q$  be two even lattices of the same rank  $n$ . Then the local conformal net  $\mathcal{A}_Q$  is the simple current extension (see [KL06, Lemma 2.1]) of  $\mathcal{A}_L$  by the subgroup  $Q/L$  of the group  $L^*/L$  of all sectors of  $\mathcal{A}_L$ .*

*Proof.* By construction it is  $\mathcal{A}_L \subset \mathcal{A}_Q$  and  $\mathcal{H}_L \subset \mathcal{H}_Q$ . Let us denote by  $\mathcal{B}$  the net obtained by the simple current extension by  $Q/L$ , which is the crossed product with automorphisms given by the adjoint action loops  $e^{if_\Delta}$  with  $\Delta_f \in Q/L$ . So clearly we can see  $\mathcal{B}$  as conformal subnet of  $\mathcal{A}_Q$  and they coincide because  $\overline{\mathcal{B}(I)\Omega} = \mathcal{H}_Q$ .  $\square$

**Remark 3.17.** *In [KL06] is given another construction of lattice models, which starts with a conformal net  $\mathcal{A}$ , which is the simple current extension of the dimension 1 sector of  $\text{Vir}_{c=1/2} \otimes \text{Vir}_{c=1/2}$ . In [KL06, Remark 2.3] the authors conjecture that  $\mathcal{A}$  is a Buchholz–Mack–Todorov extension, namely the one with  $g = 2$  which is in our language the conformal net  $\mathcal{A}_{2\mathbb{Z}}$ , where  $2\mathbb{Z}$  is the lattice with  $\langle \alpha, \beta \rangle = \alpha\beta$ . Let us assume this conjecture is true. They take even lattices  $L \supset (1/2\mathbb{Z})^n$  (coming from codes) and take the simple current extension by the group  $L/(2\mathbb{Z})^n \subset (1/2\mathbb{Z})^n/(2\mathbb{Z})^n$  of the net  $\mathcal{A}^{\otimes n}$ , which is under the conjecture isomorphic to  $\mathcal{A}_{(2\mathbb{Z})^n}$  using Remark*

3.15. By Proposition 3.16 the extended net then would be isomorphic to our net  $\mathcal{A}_L$  and the two constructions would coincide.

**3.5. Loop Group Models of Simply Laced Groups at Level 1.** We show the relation of the lattice model associated with the root lattice  $L$  of simply laced group  $G$  to the level 1 representation of the loop group of  $\mathbf{LG}$  [PS86, Seg81, Sta95].

Let  $G$  be a compact, connected, simply connected, simply laced Lie group with maximal torus  $T$ . *Simply laced* means that there is an invariant inner product on its Lie algebra  $\mathfrak{g}$  for which all roots have the same length or equivalently the Weyl group of  $G$  acts transitively on the roots. By [GF93, Theorem 3.12] the vacuum positive energy representation  $\pi$  of  $\mathbf{LG}$  at level  $k$  gives rise to a conformal net denoted by  $\mathcal{A}_{G,k}$ , defined by  $\mathcal{A}_{G,k}(I) = \pi(\mathbf{L}_I G)''$ .

Let  $\mathfrak{t}$  be the Lie algebra of  $T$  and let us identify

$$\mathfrak{t}/2\pi L \hookrightarrow T \subset G, [t] \mapsto e^t.$$

The roots of  $G$  are linear maps  $\alpha : \mathfrak{t} \rightarrow \mathbb{R}$ . For each  $\alpha$  we define a  $h_\alpha \in \mathfrak{t}$  such that  $\alpha(t) = \langle h_\alpha, t \rangle$  for  $t \in \mathfrak{t}$  where  $\langle \cdot, \cdot \rangle$  is the Cartan–Killing form which we can assume to be normalized such that  $\langle h_\alpha, h_\alpha \rangle = 2$ . This can always be realized, due to  $G$  being simply laced. In the case  $\mathbf{SU}(N)$  and  $\mathbf{Spin}(2N)$  it is given explicitly by  $\langle x, y \rangle = -\mathrm{tr}(xy)$  and  $\langle x, y \rangle = -1/2 \mathrm{tr}(xy)$ , respectively. It is well known that  $h_\alpha \in L$  and that the set of the  $h_\alpha$  with  $\alpha$  a root coincide with the  $x \in L$  such that  $\langle x, x \rangle = 2$ . By abuse of notation we identify  $\alpha$  with  $h_\alpha \in L$ , i.e.  $\langle \alpha, \beta \rangle \equiv \langle h_\alpha, h_\beta \rangle$ . We note the missing  $i$  in the exp map due to the conventions  $t^* = -t$  for  $t \in \mathfrak{t}$ , i.e. by identifying  $F = -it$  we get the relation to the former notation.

Let  $\pi$  be a positive energy representation of  $\mathbf{LT}$  with cocycle (4), where  $T$  is the torus associated with  $L$  but also a maximal torus of  $G$  by the above discussion. We note that the cocycle of the level 1 representation of  $\mathbf{LG}$  restricted to  $\mathbf{LT}$  is (equivalent to) our cocycle (4) by [PS86, Proposition 4.8.3]. More remarkable is the following result by Segal [Seg81], stating that the representation of  $\mathbf{LT}$  extends to  $\mathbf{LG}$ . This is mainly achieved by taking a limit of loops with winding number  $\Delta_f = \alpha$ , and building so called “vertex” or “bilib” operators which turn out to generate—together with the generators of loops with trivial winding number—a representation of the polynomial algebra  $L^{\mathrm{alg}} \mathfrak{g}$ , which is then exponentiated.

**Proposition 3.18** ([Seg81, Proposition 4.4]). *Let  $G$  be a compact, connected, simply connected, simply laced Lie group with maximal torus  $T$ . If  $\pi$  is a positive energy projective representation of  $\mathbf{LT}$  with cocycle above, then the action of  $\mathbf{LT}$  extends canonically to an action of  $\mathbf{LG}$ .*

Now we want to apply this result to show that certain loop group nets at level 1 are a special case of the conformal nets associated with lattices. The analog of the following result is well known in the theory of vertex operator algebras under the name Frenkel–Kac or Frenkel–Kac–Segal construction.

**Proposition 3.19** (Algebraic version of the Frenkel–Kac–Segal construction). *Let  $G$  be a compact, simple, connected, simply connected and simply laced Lie group and  $L$  its root lattice as above. Then the conformal net  $\mathcal{A}_L$  is equivalent to the loop group net  $\mathcal{A}_{G,1}$  at level 1 associated with  $\mathbf{LG}$ . In particular  $\mathcal{A}_{G,1}$  is completely rational and has  $\mu$ -index  $\mu = |L^*/L|$ .*

The case  $G = \mathbf{SU}(N)$  is stated in [Xu09, 3.1.1] and the general case in [Sta95, p. 37/38]. In principle we could try to use the result of Segal to directly prove the Proposition, but locality of the constructed exponentiated currents is not clear and has to be checked. Therefore we give a more indirect proof using an operator algebraic argument.



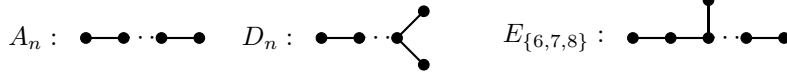


FIGURE 1. A, D, E Dynkin diagrams

*Proof.* Let  $\pi$  be the vacuum positive energy representation at level 1 of  $\mathbf{LG}$  and by Proposition 3.18 it can be assumed to act on the Hilbert space  $\mathcal{H}_L$ . We see  $\pi$  as a representation of the central extension  $\mathcal{LG}$ . It is  $\mathcal{LT} \subset \mathcal{LG}$  and in particular for every  $I \in \mathcal{I}$  also  $\mathcal{L}_I T \subset \mathcal{L}_I G$ . This implies that  $\mathcal{A}_L(I) \equiv \pi(\mathcal{L}_I T)''$  is a conformal subnet of  $\mathcal{A}_{G,1}(I) = \pi(\mathcal{L}_I G)''$ . Because  $\overline{\mathcal{A}_L(I)\Omega} = \mathcal{H}_L$  by Lemma 3.3 the two nets  $\mathcal{A}_L$  and  $\mathcal{A}_{G,1}$  have to coincide.  $\square$

**Example.** The simple, simply laced groups correspond to the Dynkin diagrams of type A, D and E (see Figure 1), namely  $\mathbf{SU}(n+1)$  for  $A_n$  with  $n \geq 1$ ,  $\mathbf{Spin}(2n)$  for  $D_n$  with  $n \geq 4$  and in the exceptional case the compact, simply connected forms of  $E_6, E_7, E_8$ . The level 1 loop group nets of these groups are therefore given by lattice models of their root lattice  $L$ , which is characterized by the basis  $\{\alpha_1, \dots, \alpha_n\}$  with  $n$  the rank of  $L$  and  $\alpha_i$  represents the  $i$ -th vertex of the Dynkin diagram. The inner product is specified by the Cartan matrix  $(C_{ij})$  via

$$\langle \alpha_i, \alpha_j \rangle = C_{ij} = \begin{cases} 2 & i = j \\ -1 & i \text{ and } j \text{ are connected by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

#### 4. BOUNDARY QUANTUM FIELD THEORY – NETS ON MINKOWSKI HALF-PLANE

In this section we want to construct local nets on Minkowski half-plane  $M_+ = \{(t, x) \in \mathbb{R}^2 : x > 0\}$  which are time-translation covariant and which we will call also simply boundary nets.

Let  $I_1, I_2$  be two intervals of the time axis such that  $I_2 > I_1$  and let us define the *double cone*

$$\mathcal{O} = I_1 \times I_2 := \{(t, x) \in \mathbb{R}^2 : t - x \in I_1, x + t \in I_2\}$$

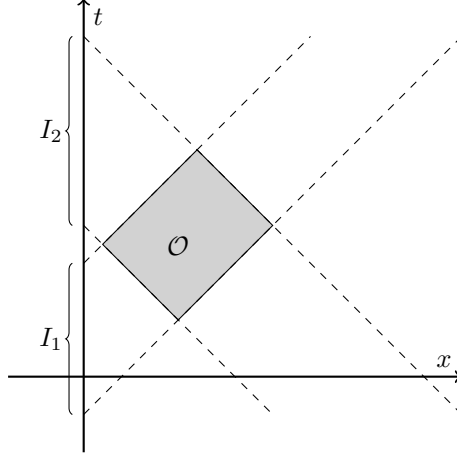
like in Figure 2. We call such a double cone  $\mathcal{O} = I_1 \times I_2$  *proper* if it has a positive distance to the boundary, i.e.  $\overline{I_1}$  and  $\overline{I_2}$  have empty intersection; the *set of proper double cones* we denote by  $\mathcal{K}_+$ .

**4.1. Local Nets of Standard Subspaces on Minkowski Half-Plane.** As an intermediate step we built up local time-translation covariant nets of standard subspaces related with the local Möbius covariant nets of standard subspaces  $H_F$  from Proposition 3.2 using the semigroup  $\mathcal{E}(H_F(0, \infty))$ .

**Definition.** By a local, time-translation covariant net of standard subspaces on  $M_+$  on a Hilbert space  $\mathcal{H}$  we mean a family  $\{K(\mathcal{O})\}_{\mathcal{O} \in \mathcal{K}_+}$  of standard subspaces of a Hilbert space  $\mathcal{H}$  which fulfills:

- A. Isotony.**  $\mathcal{O}_1 \subset \mathcal{O}_2$  implies  $K(\mathcal{O}_1) \subset K(\mathcal{O}_2)$ .
- B. Locality.** If  $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$  are space-like separated then  $K(\mathcal{O}_1) \subset K(\mathcal{O}_2)'$ .
- C. Time-translation covariance.** There is a strongly continuous one-parameter group  $U(t) = e^{itP}$  on  $\mathcal{H}$  with positive generator  $P$ , such that

$$U(t)K(\mathcal{O}) = K(\mathcal{O}_t), \quad \mathcal{O} \in \mathcal{K}_+$$

FIGURE 2. Double cone  $\mathcal{O} = I_1 \times I_2$  in  $M_+$ 

where  $\mathcal{O}_t = \mathcal{O} + (t, 0)$  is the in time-direction shifted double cone.

**Definition.** Let  $F$  be an Euclidean space and  $F_{\mathbb{C}} = F \otimes_{\mathbb{R}} \mathbb{C}$  its complexification with canonical complex conjugation  $x \mapsto \bar{x}$ . We denote by  $\mathcal{S}_F$  the space of all complex Borel functions  $\varphi : \mathbb{R} \rightarrow \mathcal{B}(F_{\mathbb{C}})$  which are boundary values of an bounded analytic function  $\mathbb{R} + i\mathbb{R}_+ \rightarrow \mathcal{B}(H)$ , i.e. for  $x, y \in F_{\mathbb{C}}$  the function  $p \mapsto (x, \varphi(p)y)$  is an analytic Borel function  $\mathbb{R} + i\mathbb{R}_+ \rightarrow \mathbb{C}$  such that  $\varphi$  is symmetric and inner, i.e. that  $\varphi(-p) = \overline{\varphi(p)}$  and  $\varphi(p) \in \mathcal{U}(F_{\mathbb{C}})$  for almost all  $p > 0$ , respectively.

We note that with  $n = \dim F$  the  $\mathcal{S}_F$  space is naturally isomorphic to  $\mathcal{S}^F$  defined in Section 2.

**Remark 4.1.** We take the standard subspace  $H_F := H_F(0, \infty)$  and  $T(t) = U(\tau(t))$  the one-parameter group of translation. Let  $\mathcal{H}_{0,F} = \mathcal{H}_0 \otimes_{\mathbb{R}} F \cong \bigoplus_{i=1}^n \mathcal{H}_0$  from Proposition 3.2, which decompose into  $n$  copies of the irreducible standard pair  $(H_0, T_0)$ , i.e.  $H_F = H_0 \otimes_{\mathbb{R}} F \cong \bigoplus_{i=1}^n H_0$ . Then  $\mathcal{E}(H_F, T)$  can be identified with  $\mathcal{S}_F$  by Theorem 2.4.

**Theorem 4.2.** Let  $H$  be a local Möbius covariant net of standard subspaces, then for each  $V \in \mathcal{E}(H(0, \infty), T)$ , with  $T(t) = U(\tau(t))$  the one-parameter group of translations, there is local, time-translation covariant net of standard subspaces on  $M_+$  given by

$$K_V : \mathcal{O} \cong I_1 \times I_2 \mapsto K_V(\mathcal{O}) := \overline{H(I_1) + VH(I_2)}.$$

*Proof.* Isotony is obvious. Locality is shown like in [LW10] and follows from  $V \in \mathcal{E}(H(0, \infty))$ . But then we have also standardness, namely  $K(\mathcal{O})$  is cyclic because  $H(I_1)$  is already cyclic and separating because  $K(\mathcal{O})'$  contains  $H(I_1^>)$  where  $I_1^>$  is the left component of the two piece complement of  $I_1$ . Time-translation covariance holds because  $V$  commutes with  $T$ .  $\square$

**Corollary 4.3.** Let  $(F, \langle \cdot, \cdot \rangle)$  be a real  $n$ -dimensional Euclidean space and  $H_F$  the net of standard subspaces from Proposition 3.2. Then for each  $V \in \mathcal{E}(H_F(0, \infty), T)$ , i.e. each element in  $\mathcal{S}_F$  as described in Remark 4.1, there is a local, time-translation covariant net of standard subspaces on  $M_+$ .

**4.2. Local Nets of von Neumann Algebras on Minkowski Half-Plane.** Let us recall the definition of a local, time-translation covariant net of von Neumann algebras  $\mathcal{A}_V$  on Minkowski half-plane [LW10] related to a local Möbius covariant net  $\mathcal{A}$  and an element of  $V$  of the semigroup  $\mathcal{E}(\mathcal{A})$ .

**Definition.** A local, time-translation covariant net (of von Neumann algebras) on Minkowski half-plane (boundary net) on a Hilbert space  $\mathcal{H}$  is a family of von Neumann algebras  $\{\mathcal{B}(\mathcal{O})\}_{\mathcal{O} \in \mathcal{K}}$  on  $\mathcal{H}$  which fulfills:

- A. Isotony.**  $\mathcal{O}_1 \subset \mathcal{O}_2$  implies  $\mathcal{B}(\mathcal{O}_1) \subset \mathcal{B}(\mathcal{O}_2)$ .
- B. Locality.** If  $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}_+$  are space-like separated then  $[\mathcal{B}(\mathcal{O}_1), \mathcal{B}(\mathcal{O}_2)] = \{0\}$ .
- C. Time-translation covariance.** There is a unitary continuous one-parameter group  $T(t) = e^{itP}$  on  $\mathcal{H}$  with positive generator  $P$ , such that

$$T(t)\mathcal{B}(\mathcal{O})T(t)^* = \mathcal{B}(\mathcal{O}_t), \quad \mathcal{O} \in \mathcal{K}_+$$

where  $\mathcal{O}_t = \mathcal{O} + (t, 0)$  is the shifted double cone.

- D. Vacuum.** There is a (up to phase) unique  $T$  invariant vector  $\Omega \in \mathcal{H}$  which is cyclic and separating for every  $\mathcal{B}(\mathcal{O})$  with  $\mathcal{O} \in \mathcal{K}_+$ .

Let  $\mathcal{A}$  be a Möbius covariant local net of von Neumann algebras on the Hilbert space  $\mathcal{H}$ , which we want regard (by restriction) as a net on  $\mathbb{R}$ . All unitaries  $V$  on  $\mathcal{H}$ , which commutes with the one-parameter group of translations  $T(t) = U(\tau(t))$ , satisfy  $V\Omega = \Omega$  and the equivalent conditions

- (1)  $V\mathcal{A}(I_2)V^*$  commutes with  $\mathcal{A}(I_1)$  for all intervals  $I_1, I_2$  of  $\mathbb{R}$  such that  $I_2 > I_1$ , i.e.  $I_2$  is contained in the future of  $I_1$ ,
- (2)  $V\mathcal{A}(a, \infty)V^* \subset \mathcal{A}(a, \infty)$  for all  $a \in \mathbb{R}$ ,
- (3)  $V\mathcal{A}(0, \infty)V^* \subset \mathcal{A}(0, \infty)$ ,

form a semigroup denoted by  $\mathcal{E}(\mathcal{A})$ . Translations  $V := T(t) \equiv U(\tau(t))$  with  $t > 0$  are elements in  $\mathcal{E}(\mathcal{A})$ . Also *internal symmetries*  $V$  of  $\mathcal{A}$ , namely  $V\mathcal{A}(I)V^* = \mathcal{A}(I)$  for all  $I \in \mathcal{I}$  give trivial examples of elements in  $\mathcal{E}(\mathcal{A})$ . Besides this trivial examples it is in general not much known if there exists other elements, but if they exists they are of the form stated as follows.

**Remark 4.4** (cf. [LW10]). Let  $\mathcal{A}$  be a conformal net, then  $\mathcal{E}(\mathcal{A}) \subset \mathcal{E}(H, T)$  with the one-parameter group of translations  $T(t) = U(\tau(t)) = e^{itP}$  and the standard subspace  $H = \overline{\mathcal{A}(0, \infty)_{\text{sa}}\Omega}$ . In particular  $H_0 := H \ominus \mathbb{R}\Omega \subset \mathcal{H}_0$  with  $\mathcal{H}_0 := \mathcal{H} \ominus \mathbb{C}\Omega$  is a non-degenerated standard pair and by Theorem 2.4 it  $V \upharpoonright_{\mathcal{H}_0} = (\varphi_{hk}(P_0))$  (by definition  $V\Omega = \Omega$ ) with  $(\varphi_{hk})$  a matrix in  $\mathcal{S}^{(\infty)}$  and  $P_0 = P \upharpoonright_{\mathcal{H}_0}$ , cf. [LW10, Corollary 2.8].

Let  $\mathcal{A}$  be a conformal net and  $V \in \mathcal{E}(\mathcal{A})$ , then we define

$$\mathcal{A}_V(\mathcal{O}) := \mathcal{A}(I_1) \vee V\mathcal{A}(I_2)V^*, \quad \mathcal{O} = I_1 \times I_2, \quad I_2 > I_1.$$

The special case  $V = 1$  is exactly the conformal boundary net  $\mathcal{A}_+$  defined in [LR04].

**Proposition 4.5** (cf. [LW10, Proposition 3.3 and Corollary 3.4]). If  $V \in \mathcal{E}(\mathcal{A})$ , then  $\mathcal{A}_V$  is a boundary net. The map  $\mathcal{E}(\mathcal{A}) \ni V \mapsto \mathcal{A}_V$  is one-to-one modulo internal symmetries, i.e.  $\mathcal{A}_{V_1} = \mathcal{A}_{V_2}$  with  $V_1, V_2 \in \mathcal{E}(\mathcal{A})$  iff  $V_1 = V_2V$  with  $V$  an internal symmetry.

The study of such boundary nets  $\mathcal{A}_V$  associated with a conformal net  $\mathcal{A}$  simplifies therefore to the study of  $\mathcal{E}(\mathcal{A})$ . So the question is the characterization and classification of the semigroup  $\mathcal{E}(\mathcal{A})$  for a given conformal net  $\mathcal{A}$ . The rest of the paper we investigate in the explicit construction of families of such elements.

**4.3. Second Quantization Boundary Nets.** Let  $(F, \langle \cdot, \cdot \rangle)$  be a  $n$ -dimensional Euclidean space. For the net  $\mathcal{A}_F$  of Abelian currents constructed in Section 3.3 we know all  $V = \Gamma(V_0) \in \mathcal{E}(\mathcal{A}_F)$  which are second quantization unitaries by the following theorem.

**Theorem 4.6** (cf. [LW10, Theorem 3.6]).  $V = \Gamma(V_0) \in \mathcal{E}(\mathcal{A}_F)$  if and only if  $V_0 = \varphi(P_0)$  with  $\varphi \in \mathcal{S}_F$ .

**Remark 4.7.** *These models are exactly the second quantization of the models constructed in Section 4.1.*

The next task is to find which  $V \in \mathcal{E}(\mathcal{A}_F)$  extend to  $\tilde{V} \in \mathcal{E}(\mathcal{A}_L)$  for an even integral lattice  $L \subset F$ .

**4.4. Semigroup for Subnets.** If we have a conformal net  $\mathcal{B}$  with conformal subnet  $\mathcal{A}$  and  $V \in \mathcal{E}(\mathcal{B})$  the question arise when  $V$  restricts to an element in  $\mathcal{E}(\mathcal{A})$ .

**Lemma 4.8.** *Let  $\Omega \in \mathcal{H}$  be a cyclic and separating vector for the von Neumann factor  $B \subset B(\mathcal{H})$  and separating for the subfactor  $A \subset B$  and we assume there is a conditional expectation  $E_A : B \rightarrow A$  which leaves the state  $\phi_\Omega = (\Omega, \cdot \Omega)$  invariant. Let  $V \in U(\mathcal{H})$  with  $V\Omega = \Omega$  and  $VBV^* \subset B$ . Then is equivalent*

- (1)  $V$  commutes with the Jones projection  $e_A$ .
- (2)  $E_A$  and  $\text{Ad}_V$  commute, i.e.  $E_A(VbV^*) = VE_A(b)V^*$  for all  $b \in B$ .

*Proof.* By definition we have  $E_A(b)e_A = e_Abe_A$  for all  $b \in B$ . Let us assume  $[e_A, V] = 0$ , then

$$\begin{aligned} VE_A(b)V^*\Omega &= VE_A(b)\Omega \\ &= VE_A(b)e_A\Omega \\ &= Ve_Abe_AV^*\Omega \\ &= e_AVbV^*e_A\Omega \\ &= E_A(VbV^*)e_A\Omega \\ &= E_A(VbV^*)\Omega \end{aligned}$$

and by the separativity of  $\Omega$  follows  $[E_A, \text{Ad}_V] = 0$ . On the other hand, let us assume now  $[E_A, \text{Ad}_V] = 0$ . Then

$$\begin{aligned} e_AVb\Omega &= e_AVbV^*e_A\Omega \\ &= E_A(\text{Ad}_V(b))e_A\Omega \\ &= \text{Ad}_V(E_A(b))e_A\Omega \\ &= VE_A(b)\Omega \\ &= VE_A(b)e_A\Omega \\ &= Ve_Abe_A\Omega \\ &= Ve_Ab\Omega \end{aligned}$$

and cyclicity implies that  $e_AV = Ve_A$ . □

**Proposition 4.9.** *Let  $\mathcal{B}$  be a conformal net on  $\mathcal{H}$  with vacuum  $\Omega$ ; let  $\mathcal{A}$  be conformal subnet of  $\mathcal{B}$  and let  $e$  be the projection on  $\overline{\mathcal{A}(I)}\Omega$  for some  $I \in \mathcal{I}$ . Further let  $V \in \mathcal{E}(\mathcal{B})$  and  $\eta = \text{Ad } V$ , then are equivalent*

- (1)  $V \upharpoonright_{e\mathcal{H}} \in \mathcal{E}(\mathcal{A})$ , regarding  $\mathcal{A}$  as a conformal net on  $e\mathcal{H}$ .
- (2) For every  $a \in \mathbb{R}$  it is  $\eta(E_a(b)) = E_a(\eta(b))$  for all  $b \in \mathcal{B}(a, \infty)$ , where  $E_a$  is the conditional expectation  $\mathcal{B}(a, \infty) \rightarrow \mathcal{A}(a, \infty)$ .
- (3) It is  $\eta(E_0(b)) = E_0(\eta(b))$  for all  $b \in \mathcal{B}(a, \infty)$ , where  $E_0$  is the conditional expectation  $\mathcal{B}(0, \infty) \rightarrow \mathcal{A}(0, \infty)$ .
- (4)  $V$  commutes with the projection  $e$ .

*Proof.* The projection  $e$  does not depend on  $I$  and is the Jones projection of the inclusion  $\mathcal{A}(I) \subset \mathcal{B}(I)$  for any  $I \in \mathcal{I}$ . Let  $V \in \mathcal{E}(\mathcal{B})$  such that  $V \upharpoonright_{e\mathcal{H}} \in \mathcal{E}(\mathcal{A})$ . We show that (4) is true, namely for  $a \in \mathcal{A}(0, \infty)$  using  $V\mathcal{A}(0, \infty)V^* \subset \mathcal{A}(0, \infty)$  we

compute

$$\begin{aligned} eVa\Omega &= eVaV^*\Omega \\ &= VaV^*\Omega \\ &= Va\Omega \\ &= Vea\Omega \end{aligned}$$

thus by continuity  $[V, e] \upharpoonright e\mathcal{H} = 0$ . Let us write  $\mathcal{H} = e\mathcal{H} \oplus e^\perp\mathcal{H}$  and let  $V_1 = eVe = Ve$  and  $V_2 = e^\perp Ve^\perp$ ; we write

$$V = \begin{pmatrix} V_1 & X \\ 0 & V_2 \end{pmatrix}.$$

Because  $V$  and  $V_1$  are unitaries on  $\mathcal{H}$  and  $e\mathcal{H}$ , respectively, it follows that  $X = 0$ . We claim that also  $[V, e] \upharpoonright e^\perp\mathcal{H} = 0$ , namely for  $\xi \in e^\perp\mathcal{H}$  we calculate

$$e_AV\xi = e_AV_2\xi = e_Ae_A^\perp Ve_A^\perp\xi = 0 = Ve_A\xi.$$

Conversely, let  $V \in \mathcal{E}(\mathcal{A})$  with  $[e, V] = 0$ . By Lemma 4.8  $\eta = \text{Ad}_V$  commutes with the conditional expectation  $E : \mathcal{B}(0, \infty) \rightarrow \mathcal{A}(0, \infty)$ , i.e.  $E(VaV^*) = VE(a)V^*$  for  $a \in \mathcal{A}(0, \infty)$ . We claim that  $\text{Ad}_V$  is an endomorphism of  $\mathcal{A}(0, \infty)$ , namely

$$\begin{aligned} V\mathcal{A}(0, \infty)V^* &= VE(\mathcal{B}(0, \infty))V^* \\ &= E(V\mathcal{B}(0, \infty)V^*) \\ &\subset E(\mathcal{B}(0, \infty)) \\ &= \mathcal{A}(0, \infty). \end{aligned}$$

Since  $V$  and  $e$  commute  $V \upharpoonright_{e\mathcal{H}} = eVe$  is a unitary on  $e\mathcal{H}$  and commutes with  $T(t) \upharpoonright_{e\mathcal{H}} = eT(t)e$ , i.e.  $V \upharpoonright_{e\mathcal{H}} \in \mathcal{E}(\mathcal{A})$ .  $\square$

**4.5. Extensions for the Crossed Product with Free Abelian Groups.** Let  $\mathcal{M}$  be a type III factor.  $\text{End}(\mathcal{M})$  is a tensor- $C^*$ -category with objects  $\rho \in \text{End}(\mathcal{M})$  normal endomorphisms of  $\mathcal{M}$  and arrows  $\text{Hom}_{\mathcal{M}}(\rho, \eta) = \{t \in \mathcal{M} : t\rho(x) = \eta(x)t \text{ for all } x \in \mathcal{M}\}$ . For any  $\rho \in \text{End}(\mathcal{M})$  we have  $\text{Hom}(\rho, \rho) \ni \text{id}_\rho := 1$ . The tensor product is defined by the composition  $\eta \otimes \rho := \eta\rho$  and for  $f \in \text{Hom}_{\mathcal{M}}(\rho, \rho')$  and  $g \in \text{Hom}_{\mathcal{M}}(\eta, \eta')$  it is  $f \otimes g := f\rho(g) = \rho'(g)f \in \text{Hom}_{\mathcal{M}}(\rho\eta, \rho'\eta')$ .

Let  $L$  be a free Abelian group of rank  $n$  with generators ( $\mathbb{Z}$ -basis)  $\{\alpha_1, \dots, \alpha_n\}$  and let  $\beta$  be a faithful action on a von Neumann algebra  $\mathcal{M} \subset B(\mathcal{H}_0)$  with cyclic and separating vector  $\Omega$ . The action of  $L$  is characterized by the action of the automorphisms  $\beta_i := \beta_{\alpha_i}$ . Let  $\mathcal{H} = \bigoplus_{\alpha \in L} \mathcal{H}_\alpha \supset \mathcal{H}_0$ . We assume  $\beta_i$  to be implemented by unitaries  $U_i$  mapping  $\mathcal{H}_\alpha \rightarrow \mathcal{H}_{\alpha+\alpha_i}$ . We note that

$$L \ni g = \sum_{i=1}^n g_i \alpha_i \mapsto U_g = U_1^{g_1} \cdots U_n^{g_n} \quad g_i \in \mathbb{Z}$$

defines a projective representation of  $L$  on  $\mathcal{H}$ .

Let  $\tilde{\mathcal{M}} = \mathcal{M} \rtimes_\beta Q \subset B(\mathcal{H})$  be the von Neumann algebra generated by  $\mathcal{M}$  and  $\{U_i\}$  on  $\mathcal{H}$ . We are interested in extension of endomorphisms of  $\mathcal{M}$  to endomorphisms of  $\tilde{\mathcal{M}}$ . The following is in principal a generalization of [LW10, Proposition 3.8 and 3.9].

**Lemma 4.10.** *Let  $\mathcal{M}$  as above and  $\tilde{\mathcal{M}}_0 \subset \tilde{\mathcal{M}}$  the algebra finitely generated by  $\mathcal{M}$  and  $\{U_\alpha\}_{\alpha \in L}$ . Further let  $R : L \rightarrow L$  be an automorphism of  $L$  and for  $i = 1, \dots, n$  let  $\tilde{\beta}_i := \beta_{R(\alpha_i)}$  be automorphisms of  $\mathcal{M}$  having  $\tilde{U}_i = U_{R(\alpha_i)}$  as implementing unitaries. If there exist unitaries  $z_i \in \text{Hom}_{\mathcal{M}}(\tilde{\beta}_i \circ \eta, \eta \circ \beta_i)$  satisfying*

$$z_i \tilde{\beta}_i(z_j) = z_j \tilde{\beta}_j(z_i)$$

then  $\eta$  extends to an endomorphism  $\tilde{\eta}_0$  of  $\tilde{\mathcal{M}}_0$  characterized by  $\tilde{\eta}_0(U_i) = z_i \tilde{U}_i$ .

*Proof.* Each  $g \in Q$  can uniquely be written as  $g = \sum_i g_i \alpha_i$  with  $g_i \in \mathbb{Z}$  for  $i = 1, \dots, n$ . Further we denote  $U_g := U_1^{g_1} \dots U_n^{g_n}$  which defines a projective representation of  $Q$ . For finite non-zero  $a_g \in M$  we define:

$$\tilde{\eta}_0 : \sum_g a_g U_g \mapsto \sum_g a_g (z_1 U_1)^{g_1} \dots (z_n U_n)^{g_n} =: \sum_g a_g v_g U_g$$

which is well-defined because the action is faithful. It is easy to check, that  $\tilde{\eta}_0$  is an endomorphism if  $v_g \in M$  is a “cocycle” (similar like in [Kaw01]) satisfying

$$\begin{aligned} v_g \tilde{\beta}_g(\eta(x)) &= \eta(\beta_g(x)) v_g & x \in \mathcal{M} \\ v_{g+h} &= v_g \tilde{\beta}_g(v_h) \end{aligned}$$

with  $\tilde{\beta}_g = \text{Ad } \tilde{U}_g$ . Indeed, using the tensor category calculus we write for arrows  $t : \sigma\eta \rightarrow \eta\sigma'$  and  $s : \rho\eta \rightarrow \eta\rho'$

$$s \diamond t := (s \otimes \text{id}_{\tau'}) (\text{id}_\rho \otimes t) : \rho\sigma\eta \xrightarrow{\text{id}_\rho \otimes t} \rho\eta\sigma' \xrightarrow{s \otimes \text{id}_{\tau'}} \eta\rho'\tau' \quad (6)$$

for example  $z_i \diamond z_j := (z_i \otimes \text{id}_{\beta_j}) (\text{id}_{\tilde{\beta}_i} \otimes z_j) \equiv z_i \tilde{\beta}_i(z_j)$ . The condition  $z_i \tilde{\beta}_i(z_j) = z_j \tilde{\beta}_j(z_i)$  reads  $z_i \diamond z_j = z_j \diamond z_i$ . Let us write  $z_i^- = \tilde{\beta}_i^{-1}(z_i^*)$  in particular  $z_i \diamond z_i^- = z_i^- \diamond z_i = 1$  and we have also

$$\begin{aligned} z_j \diamond z_i^- &= z_j \tilde{\beta}_j(\tilde{\beta}_i^{-1}(z_i^*)) \\ &= \tilde{\beta}_i^{-1}(\tilde{\beta}_i(z_j) \tilde{\beta}_j(z_i^*)) \\ &= \tilde{\beta}_i^{-1}(z_i^* z_i \tilde{\beta}_i(z_j) \tilde{\beta}_j(z_i^*)) \\ &= \tilde{\beta}_i^{-1}(z_i^* z_j \tilde{\beta}_j(z_i) \tilde{\beta}_j(z_i^*)) \\ &= \tilde{\beta}_i^{-1}(z_i^* z_j) \\ &= z_i^- \diamond z_j. \end{aligned}$$

With this notation it is

$$v_g = z_1^{\diamond g_1} \diamond \dots \diamond z_n^{\diamond g_n} := \underbrace{z_1^\pm \diamond \dots \diamond z_1^\pm}_{\pm g_1\text{-times}} \diamond \dots \diamond \underbrace{z_n^\pm \diamond \dots \diamond z_n^\pm}_{\pm g_n\text{-times}} \in \text{Hom}_{\mathcal{M}}(\tilde{\beta}_g \eta, \eta \beta_g)$$

which does not depend on the order of the  $z_i$  and  $z_i^-$ , so in particular

$$v_g \tilde{\beta}_g(v_h) \equiv (v_g \otimes \text{id}_{\beta_h}) (\text{id}_{\tilde{\beta}_g} \otimes v_h) = v_g \diamond v_h = v_{g+h}. \quad \square$$

**Proposition 4.11.** *Let  $\eta$  be a  $\phi_\Omega$ -preserving endomorphism of  $\mathcal{M}$ . Under the hypothesis of Lemma 4.10 the endomorphism  $\eta$  extends to a  $\phi_\Omega$ -preserving endomorphism  $\tilde{\eta}$  of  $\tilde{\mathcal{M}}$  characterized by  $\tilde{\eta}(U_i) = z_i \tilde{U}_i$ .*

*Proof.*  $\tilde{\eta}_0$  preserves the conditional expectation  $\sum a_\alpha U_\alpha \mapsto a_0$  so it preserves the state  $\phi_\Omega$  and  $\Omega$  is cyclic for  $\tilde{\eta}_0(\tilde{\mathcal{M}}_0)$ , because the space  $\tilde{\eta}_0(\tilde{\mathcal{M}}_0)\Omega$  contains  $\mathcal{H}_0$  and is  $U_i$  invariant. Finally, there exists a unitary  $\tilde{V}$  with  $\tilde{V}x\Omega = \tilde{\eta}_0(x)\Omega$  and  $\tilde{\eta} = \text{Ad } \tilde{V}$  is the extension.  $\square$

Let us in the case  $\eta \in \text{Aut}(\mathcal{M})$  and  $v_g \in \mathbb{T}$  speak of a *internal symmetry*. In the special case  $z_i = 1$  for  $i = 1, \dots, n$  it is  $\tilde{\beta}_i \eta = \eta \beta_i$  and  $\eta$  extends to a symmetry  $\tilde{\eta}$  related to the automorphism  $R$  of  $L$ ; in the case  $\eta = \text{id}_{\mathcal{M}}$  we speak of a *toral symmetry*. On the other hand lets in the case  $R = \text{id}_L$  talk about *charge preserving* endomorphisms. A charge preserving internal symmetry is toral.

**Remark 4.12.** Let  $\tilde{\tau} : U_i \mapsto z_i U_i$  a charge preserving transformation which extends  $\tau$  and  $\tilde{\sigma} : U_i \rightarrow c_i \tilde{U}_i$  inner then  $\tilde{\tau}\tilde{\sigma} : U_i \mapsto c_i z_i \tilde{U}_i$  defines an extension of  $\tau\sigma$ .

Given  $\tilde{\eta}$  an extension of  $\eta$  with  $R$  and  $\tilde{\sigma}$  an inner transformation with  $\tilde{\sigma} : \tilde{U}_i \rightarrow c_i U_i^*$  where  $c_i \in \mathbb{T}$  extending some  $\sigma \in \text{Aut}(\mathcal{M})$  having  $R^{-1} : L \rightarrow L$  as automorphism of  $L$ . Then  $\tilde{\eta}\tilde{\tau}$  is charge preserving.

**Remark 4.13.** In the case when  $\eta$  has a charge preserving extension  $\tilde{\eta} : U_i \mapsto z_i U_i$ , let us look into the full monoidal subcategory  $\mathcal{C}$  generated by  $\beta_i$  and  $\eta$ . Then  $v_g \in \text{Hom}_{\mathcal{M}}(\beta_g \eta, \eta \beta_g)$  is similar (the number of endomorphisms is not finite) two the half-braiding with respect to  $\eta$  defined in [Izu00]. The condition  $z_i \beta_i(z_j) = z_j \beta_j(z_i)$  reflects the fusion-braid equation.

We also have a converse of Proposition 4.11, namely that extensions of this form are given by  $z_i$  like in Lemma 4.10.

**Proposition 4.14.** If  $\tilde{\eta}$  is an endomorphism of  $\tilde{\mathcal{M}}$  and restricts to an endomorphism of  $\mathcal{M}$  and  $\eta(e_\alpha) = e_{R(\alpha)}$  such that  $U_i U_j = \hat{c}(\alpha_i, \alpha_j) U_j U_i \iff \tilde{U}_i \tilde{U}_j = \hat{c}(\alpha_i, \alpha_j) \tilde{U}_j \tilde{U}_i$ . Then there exist  $z_i \in \text{Hom}_{\mathcal{M}}(\tilde{\beta}_i \eta, \eta \beta_i)$  with  $z_i \tilde{\beta}_i(z_j) = z_j \tilde{\beta}_j(z_i)$ .

*Proof.* If  $\tilde{\eta}$  restricts to an endomorphism of  $\mathcal{M}$  means it commutes with the Jones projection  $e_0$  by Proposition 4.9 and  $z_i := \tilde{\eta}(U_i) \tilde{U}_i^* \in \tilde{\mathcal{M}}$ . But also  $z_i \in \mathcal{M}$  because it commutes with  $e_0$ . Finally

$$\begin{aligned} z_i \tilde{\beta}_i(\eta(x)) &= \tilde{\eta}(U_i) \eta(x) \tilde{U}_i^* \\ &= \eta(\beta_i(x)) \tilde{\eta}(U_i) \tilde{U}_i^* \\ &= \eta(\beta_i(x)) z_i \end{aligned}$$

and

$$\begin{aligned} z_i \tilde{\beta}_i(z_j) &= \tilde{\eta}(U_i U_j) \tilde{U}_j^* \tilde{U}_i^* \\ &= \tilde{\eta}(\hat{c}(\alpha_i, \alpha_j) U_j U_i) \hat{c}(\alpha_i, \alpha_j)^* U_i^* \tilde{U}_j^* \\ &= \tilde{\eta}(U_j U_i) U_i^* \tilde{U}_j^* \\ &= z_j \tilde{\beta}_j(z_i) \end{aligned}$$

which completes the proof.  $\square$

**Proposition 4.15.** Let  $\mathcal{A}$  be conformal net and  $\mathcal{A}_{\text{ext}}$  a local extension by  $\{\beta_i\}_{i=1\dots n}$  automorphisms of  $\mathcal{A}$  localized in  $(0, \infty)$ , such that  $\mathcal{A}_{\text{ext}}(0, \infty) = \mathcal{A}(0, \infty) \rtimes_{\beta} Q$ . Further let  $V \in \mathcal{E}(\mathcal{A})$ ,  $\eta = \text{Ad } V$ . Then there exist an extension  $\tilde{V}$  of  $V$ , with  $\tilde{V} \in \mathcal{E}(\mathcal{A}_{\text{ext}})$  associated to an automorphism  $R : \alpha_i \mapsto \tilde{\alpha}_i$  of  $L$  if and only if

(1) there are  $z_i \in \text{Hom}_{\mathcal{A}(0, \infty)}(\tilde{\beta}_i \eta, \eta \beta_i)$  for  $i = 1, \dots, n$  such that

$$z_i \tilde{\beta}_i(z_j) = z_j \tilde{\beta}_j(z_i),$$

(2) and there are unitary one-parameter groups  $u_i(t)$  with  $\text{Ad } u_i(t) \beta_i(\tau_t(x)) = \tau_t(\beta_i(x))$  for all  $x \in \mathcal{A}(0, \infty)$  with  $u_i(t) \beta_i(u_j(t)) = u_j(t) \beta_j(u_i(t))$  which extends  $\tau_t = \text{Ad } T(t)$  from  $\mathcal{A}$  to  $\mathcal{A}_{\text{ext}}$  by  $\tilde{\tau}_t(U_i) = u_i(t) U_i$  satisfying

$$z_i \tilde{u}_i(t)^* = \eta(u_i(t)^*) \tau_t(z_i).$$

*Proof.* The first part follows directly by Proposition 4.11 and the converse by Proposition 4.14. We note that  $\tau_t$  is extended to  $\mathcal{A}_{\text{ext}}$  via a cocycle  $u_i(t)$  namely  $\tilde{\tau}_t(U_i) = u_i(t) U_i$ . That  $\tilde{\tau}_t$  commutes with  $\tilde{\eta}$  means equality of

$$\begin{aligned} \tilde{\tau}_t(\tilde{\eta}(U_i)) &= \tilde{\tau}_t(z_i \tilde{U}_i) = \tau_t(z_i) \tilde{u}_i(t) \tilde{U}_i \\ \tilde{\eta}(\tilde{\tau}_t(U_i)) &= \tilde{\eta}(u_i(t) U_i) = \eta(u_i(t)) z_i \tilde{U}_i \end{aligned}$$

which is equivalent with

$$\eta(u_i(t)^*) \tau_t(z_i) = z_i \tilde{u}_i(t)^*.$$

□

**4.6. Boundary Nets Associated with Lattices.** We investigate in the semi-group elements for the conformal nets associated with lattices and give corresponding boundary nets. Here it is more convenient to use the real line picture by identifying  $x = \tan \theta/2$ . For  $f \in L^2(\mathbb{R}, F)$ , we denote its Fourier transform by  $\hat{f} \in L^2(\mathbb{R}, F_{\mathbb{C}})$ , namely:

$$\begin{aligned} \hat{f}(p) &= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ipx} f(x) dx & \overline{\hat{f}(p)} &= \hat{f}(-p) \\ f(x) &= \int_{\mathbb{R}} e^{ipx} \hat{f}(p) dp. \end{aligned}$$

We note that in  $\mathcal{H}_{0,F}$  the complex structure is given by  $\widehat{\mathcal{T}f}(p) = -i \operatorname{sign}(p) \hat{f}(p)$  and the action of the translation by  $T(t) = e^{itP}$  by

$$\widehat{T(t)f}(p) = e^{-i \operatorname{sign}(p)t|p|} \hat{f}(p) = e^{-itp} \hat{f}(p).$$

and the sesquilinear form by

$$\omega(f, g) := \operatorname{Im}(f, g) = \frac{1}{2} \int_{\mathbb{R}} \langle f'(x), g(x) \rangle \frac{dx}{2\pi} =: \frac{1}{2} \int \langle f', g \rangle.$$

The norm of  $\mathcal{H}_{0,F}$  is  $\|f\|_{\mathcal{H}_{0,F}} = \operatorname{const.} \int_0^\infty \|\hat{f}(p)\|_{F_{\mathbb{C}}} p dp$  and we note that  $f \in L^2(\mathbb{R}, F)$  is in  $\mathcal{H}_{0,F}$  if the norm  $\|f\|_{\mathcal{H}_{0,F}}$  is finite.

Let  $L$  be an even lattice. We write it as a sum of irreducible components  $L = L_1 \oplus \dots \oplus L_k$  with  $\langle L_i, L_j \rangle = 0$ . We call a linear, isometric, isomorphic map  $L \xrightarrow{\sim} L$  an *automorphism* of  $L$  and denote the set of automorphisms of  $L$  by  $\operatorname{Aut} L$ .

**Definition.** Let  $R : L \xrightarrow{\sim} L$  be an automorphism of  $L = L_1 \oplus \dots \oplus L_k$  and  $F = L \otimes_{\mathbb{Z}} \mathbb{R}$ . We denote by  $\mathcal{S}_{L,R}$  the space of elements  $\varphi \in \mathcal{S}_F$ , such that  $\varphi(p)$  maps  $\mathbb{C}\alpha_i$  to  $\mathbb{C}R\alpha_i$  for all  $i = 1, \dots, n$  and for almost all  $p$ .

**Lemma 4.16.** *With this notation, there is a bijection between  $\mathcal{S}_{L,R}$  and  $\mathcal{S}^{\times k}$ . It is given by  $\mathcal{S}^{\times k} \ni (\varphi_1, \dots, \varphi_k) \mapsto \varphi$  with*

$$\varphi(p)\alpha_i := \varphi_j(p)R\alpha_i \quad \alpha_i \in L_j.$$

*Proof.* We write  $\varphi(p)\alpha_i = c_i(p)R\alpha_i$  with  $c_i \in \mathcal{S}$ . That  $\varphi(p) \in \operatorname{U}(F_{\mathbb{C}})$  is equivalent with,  $\overline{c_i(p)}c_j(p)\langle \alpha_i, \alpha_j \rangle = \langle \alpha_i, \alpha_j \rangle$  for all  $i, j$ . But this means  $c_i(p) = c_j(p)$  on each component. □

Let us abbreviate  $\tilde{\alpha}_i = R\alpha_i$ . We call  $\varphi \in \mathcal{S}$  *Hölder continuous* at 0, if

$$p \mapsto \frac{|\varphi(p) - 1|^2}{|p|}$$

is locally integrable and denote the subset of Hölder continuous functions by  $\mathcal{S}^{\operatorname{Hö}l}$ . In an obvious way we denote  $\mathcal{S}_{L,R}^{\operatorname{Hö}l} \cong \mathcal{S}^{\operatorname{Hö}l \times k}$ .

**Lemma 4.17.** *Let  $L$  be an even lattice,  $R \in \operatorname{Aut}(L)$  and  $F = L \otimes_{\mathbb{Z}} \mathbb{C}$ . Let  $\eta = \operatorname{Ad} V$  with  $V \in \mathcal{E}(\mathcal{A}_F)$  related to  $\varphi \in \mathcal{S}_{L,R}^{\operatorname{Hö}l} \subset \mathcal{S}_F$  like in Theorem 4.6. Then there exist unitaries  $z_i \in \mathcal{A}_F(0, \infty)$ , such that*

- (1)  $z_i \in \operatorname{Hom}_{\mathcal{A}_F(0, \infty)}(\tilde{\beta}_i \eta, \eta \beta_i)$ ,
- (2)  $z_i \tilde{\beta}_i(z_j) = z_j \tilde{\beta}_j(z_i)$ ,
- (3)  $z_i \tilde{u}_i(t)^* = \eta(u_i(t)^*) \tau_t(z_i)$ .



*Proof.* The automorphisms localized in  $(0, \infty)$  can be chosen to be

$$\beta_i(W(f)) = e^{-i \int \langle f, m \cdot \alpha_i \rangle} W(f)$$

with  $m : \mathbb{R} \rightarrow \mathbb{R}$  a Schwartz function with support in  $(0, \infty)$  and  $\int_{\mathbb{R}} m(x) = 1$ . Let  $R \in \text{Aut}(L)$  and  $\varphi \in \mathcal{S}_{L,R}^{\text{Höl}}$  with corresponding  $(\varphi_1, \dots, \varphi_k) \in \mathcal{S}^{\text{Höl} \times k}$  given by Lemma 4.16 and let us formally define  $m_i := \varphi_j(P)m\tilde{\alpha}_j$  for  $\alpha_i \in L_j$ , more precisely

$$m_i(x) = \int e^{ipx} \overline{\varphi_j(p)} \hat{m}(p) \tilde{\alpha}_i dp \quad \alpha_i \in L_j.$$

Then  $m\tilde{\alpha}_i - m_i$  has zero integral, because  $\hat{m}(0)\tilde{\alpha}_i = \hat{m}_i(0)$  and it is in  $H_F(0, \infty)$  because  $\varphi_j \in \mathcal{S}$  is analytic in the upper strip using the Paley-Wiener theorem. Further its principal  $M_i - M\tilde{\alpha}_i$  has support in  $(0, \infty)$  and is in  $\mathcal{H}_{0,F}$  because the norm

$$\int_0^\infty \|\hat{M}_i(p) - \hat{M}(p)\tilde{\alpha}_i\|_{F_C}^2 p dp = \int_0^\infty \frac{|\varphi(p) - 1|^2}{|p|} \|\hat{m}(p)\tilde{\alpha}_i\|_{F_C}^2 dp < \infty$$

is finite due to the Hölder continuity. In particular, we get  $M_i - M\tilde{\alpha}_i \in H_F(0, \infty)$ .

We claim that  $z_i := W(M_i - M\tilde{\alpha}_i) \in \mathcal{A}_F(0, \infty)$  defines unitaries with the wanted properties. Namely, to check (1) let us calculate

$$\begin{aligned} \text{Ad } z_i(\tilde{\beta}_i(\eta(W(f)))) &= \text{Ad } z_i(\tilde{\beta}_i(W(V_0 f))) \\ &= e^{-i \int \langle m\tilde{\alpha}_i, V_0 f \rangle} \text{Ad } z_i(W(V_0 f)) \\ &= e^{-i \int \langle m\tilde{\alpha}_i, V_0 f \rangle} e^{-i \int \langle (M_i - M\tilde{\alpha}_i)', V_0 f \rangle} W(V_0 f) \\ &= e^{-i \int \langle (V_0 M\alpha_i)', V_0 f \rangle} W(V_0 f) \\ &= e^{-i \int \langle m \cdot \alpha_i, f \rangle} W(V_0 f) \\ &= e^{-i \int \langle m \cdot \alpha_i, f \rangle} \eta(W(f)) \\ &= \eta(\beta_i(W(f))). \end{aligned}$$

To verify (2) we compute

$$\begin{aligned} z_i \tilde{\beta}_i(z_j) &= W(M_i - M\tilde{\alpha}_i) \tilde{\beta}_i(W(M_j - M\alpha_j)) \\ &= e^{-i \int \langle m \cdot \tilde{\alpha}_i, M_j - M\tilde{\alpha}_j \rangle} W(M_i - M\tilde{\alpha}_i) W(M_j - M\tilde{\alpha}_j) \\ &= e^{-i \int \langle m\tilde{\alpha}_i, M_j - M\alpha_j \rangle} e^{-\frac{i}{2} \int \langle M'_i - m\tilde{\alpha}_i, M_j - M\tilde{\alpha}_j \rangle} W(M_i + M_j - M(\tilde{\alpha}_i + \tilde{\alpha}_j)) \end{aligned}$$

which is symmetric under  $i \leftrightarrow j$  realizing that

$$\begin{aligned} \langle M'_i + m\tilde{\alpha}_i, M_j - M\tilde{\alpha}_j \rangle &= \langle M'_i, M_j \rangle - \langle M'_i, M\tilde{\alpha}_j \rangle + \langle m\tilde{\alpha}_i, M_j \rangle - \langle m\tilde{\alpha}_i, M\alpha_j \rangle \\ &= \langle m\tilde{\alpha}_i, M_j \rangle + \langle m\tilde{\alpha}_j, M_i \rangle - \langle M_i, M\tilde{\alpha}_j \rangle' + \frac{1}{2} \langle M_i, M_j \rangle' - \frac{1}{2} \langle M\alpha_i, M\alpha_j \rangle' \end{aligned}$$

and noting that  $\langle M_i, M\tilde{\alpha}_j \rangle = \langle M_j, M\tilde{\alpha}_i \rangle$ . This is true, because if  $\langle \tilde{\alpha}_i, \tilde{\alpha}_j \rangle \neq 0$ , then  $\alpha_i$  and  $\alpha_j$  are connected and sit in the same component, e.g.  $L_k$  and  $M_i$  and  $M_j$  are obtained both by multiplication with the same function  $\varphi_k$ .

$$\begin{aligned}
& \text{To show (3) we recall that } u_i(t) = W((M_t - M)\alpha_i) \text{ and } \tilde{u}_i(t) = W((M_t - M)\tilde{\alpha}_i) \\
& z_i \tilde{u}_i(t)^* = W(M_i - M\tilde{\alpha}_i) \tilde{c}_i W(M\tilde{\alpha}_i - M_t \tilde{\alpha}_i) \\
& = \tilde{c}_i^* e^{-i/2 \int \langle M'_i, M\tilde{\alpha}_i \rangle - \langle M'_i, M_t \tilde{\alpha}_i \rangle - \langle m\tilde{\alpha}_i, M\tilde{\alpha}_i \rangle + \langle m\tilde{\alpha}_i, M_t \tilde{\alpha}_i \rangle} W(M_i - M_t \tilde{\alpha}_i) \\
& = \tilde{c}_i^* e^{-i/2 \int \langle M'_{ti}, M_t \tilde{\alpha}_i \rangle - \langle M'_i, M_t \tilde{\alpha}_i \rangle - \langle M'_{ti}, M_{ti} \rangle + \langle M'_i, M_{ti} \rangle} W(M_i - M_t \tilde{\alpha}_i) \\
& = W(M_i - M_{ti}) W(M_{ti} - M_t \tilde{\alpha}_i) \\
& = \eta(W(M\alpha_i - M_t \alpha_i)) \tau_t(W(M_i - M\tilde{\alpha}_i)) \\
& = \eta(u_i(t)^*) \tau_t(z_i)
\end{aligned}$$

where  $M_i$  as before,  $M_t(x) := M(x - t)$  and  $M_{it}(x) := M_i(x - t)$ .  $\square$

**Remark 4.18.** In particular, the Theorem shows that  $V \in \mathcal{E}(\mathcal{A}_F)$  corresponding to  $\mathcal{S}_{L,R}^{\text{Hö}l}$  extends to  $\tilde{V} \in \mathcal{E}(\mathcal{A}_L)$  by Proposition 4.15. For the case of boundary nets we can choose  $R = \text{id}_L$  because the obtained  $\tilde{V}$  just differ by internal symmetries.

Putting this together we have proven.

**Proposition 4.19.** Let  $L = L_1 \oplus \dots \oplus L_k$  be an even integral lattice with  $k$  components and  $\varphi \in \mathcal{S}_{L,1}^{\text{Hö}l}$  corresponding to  $(\varphi_1, \dots, \varphi_k) \in \mathcal{S}^{\text{Hö}l \times k}$ , then there is a local, time-translation covariant net on Minkowski half-plane associated with the conformal net  $\mathcal{A}_L$  and  $\varphi$ .

**Corollary 4.20.** Let  $G$  be a compact, simple, connected, simply connected, simply laced Lie group and  $\varphi \in \mathcal{S}^{\text{Hö}l}$ , then there is a local, time-translation covariant net on Minkowski half-plane associated with the conformal net  $\mathcal{A}_{G,1}$  (associated with the level 1 representation of  $G$ ) and  $\varphi$ . Further if  $G$  is just semisimple, i.e. it is a product of  $k$  simple groups of type  $A$ ,  $D$  and  $E$ , then we obtain a such net for every  $(\varphi_1, \dots, \varphi_k) \in \mathcal{S}^{\text{Hö}l \times k}$ .

**4.7. Further Examples Coming from the Orbifold Construction.** In this section we want to give further examples of boundary nets coming from the loop group net of  $G = \text{Spin}(2n)$  at level 2 using the orbifold construction.

**Definition.** Let  $\mathcal{A}$  be a conformal net on  $\mathcal{H}$ . Let  $V : G \rightarrow \text{U}(\mathcal{H})$  be a faithful unitary representation of a finite group  $G$  on  $\mathcal{H}$ . It is said that  $G$  acts properly on the conformal net  $\mathcal{A}$  if the following conditions are satisfied:

- (1) for each  $I \in \mathcal{I}$  and each  $g \in G$ ,  $\alpha_g(a) := V(g)aV(g)^* \in \mathcal{A}(I)$  for all  $a \in \mathcal{A}(I)$ ,
- (2) for each  $g \in G$  it is  $V(g)\Omega = \Omega$ .

**Definition.** Let  $\mathcal{A}$  be a conformal net on  $\mathcal{H}$  and let  $V : G \rightarrow \text{U}(\mathcal{H})$  be a proper action on  $\mathcal{H}$ . Let  $\mathcal{H}_0 = \{x \in \mathcal{H} : V(g)x = x \text{ for all } g \in G\}$  and  $P_0$  the projection on  $\mathcal{H}_0$ . Then  $\mathcal{B}(I) = \{a \in \mathcal{A}(I) : \text{Ad } V(g)a = a\}$  is a conformal subnet and we denote by  $\mathcal{A}^G(I) = \mathcal{B}(I)P_0$  the conformal net on  $\mathcal{H}_0$ , called the orbifold net.

We use following result from [Xu00] to obtain loop group net of  $\text{L Spin}(m)$  at level 2. By identifying  $\mathbb{R}^{2m} \ni (x, y) \mapsto x + iy \in \mathbb{C}^m$  where  $x, y$  are “column” vectors with  $m$  real entries we have the natural inclusion  $\text{L SU}(m)_1 \times \text{LU}(1)_m \subset \text{L Spin}(2m)_1$  where  $\text{U}(1)$  acts on  $\mathbb{C}^m$  as scalars. A further natural inclusion is given by  $\text{L Spin}(m)_2 \subset \text{L SU}(m)_1 \subset \text{L Spin}(2m)_1$ . Let  $K := (I_m, -I_m) \in \text{SO}(2m)$  which lifts to  $\text{Spin}(2m)$ . Then it is  $KAK = \bar{A}$  for  $A \in \text{SU}(m)$  and  $KAK = A$  for  $A \in \text{Spin}(2m)$ .  $K$  defines a proper action of  $\mathbb{Z}_2$  on  $\mathcal{A}_{(\text{SU}(n),1)}$ .

**Proposition 4.21** (Lemma 5.1 [Xu00]). The loop group net  $\mathcal{A}_{(\text{Spin}(m),2)}$  of  $\text{Spin}(m)$  at level 2 is isomorphic to the  $\mathbb{Z}_2$  orbifold net  $\mathcal{A}_{(\text{SU}(m),1)}^{\mathbb{Z}_2}$  of the level 1 loop group net  $\mathcal{A}_{(\text{SU}(n),1)}$  associated with  $\text{L SU}(2)$ , i.e.  $\mathcal{A}_{(\text{Spin}(m),2)} \cong \mathcal{A}_{(\text{SU}(m),1)}^{\mathbb{Z}_2} \cong \mathcal{A}_{A_{m-1}}^{\mathbb{Z}_2}$ .

**Proposition 4.22.** *Let  $\varphi \in \mathcal{S}^{\text{H\"ol}}$ , then there is a local, time translation covariant net on Minkowski half-plane associated with the loop group net  $\mathcal{A}_{\text{Spin}(m),2}$  of  $\text{Spin}(m)$  at level 2.*

*Proof.* Let  $L$  be the  $A_{n-1}$  lattice,  $F = L \otimes_{\mathbb{Z}} \mathbb{R}$  the associated Euclidean space and  $\eta = \text{Ad } V$  the endomorphism  $\mathcal{A}_F$  associated with the function  $\varphi(p) \cdot 1_{n-1}$ . We choose the special cocycle

$$z_i = e^{-i \int \langle m\alpha_i, M_i - M\alpha_i \rangle} W(M_i - M\alpha_i) =: \beta_i^{1/2}(W(M_i - M\alpha_i))$$

similar like before which differs from the  $z_i$  just by a phase and denote  $\tilde{\eta} = \text{Ad } \tilde{V}$  the endomorphism of  $\mathcal{A}_L \equiv \mathcal{A}_{\text{SU}(n),1}$  coming from the cocycle  $z_i$ . Let  $\sigma : W(f) \mapsto W(-f)$ ,  $U_\alpha \mapsto cU_\alpha^*$ . This gives a proper action of  $\mathbb{Z}_2$ . Finally  $\eta$  and  $\sigma$  commute

$$\begin{aligned} \eta(\tau(U_i)) &= \eta(c_i U_i^*) \\ &= \beta_i^{-1/2}(W(M_i - M\alpha_i)^*) c_i U_i^* \\ &= \tau(\beta_i^{1/2}(W(M_i - M\alpha_i)) U_i) \\ &= \tau(z_i U_i) \\ &= \tau(\eta(U_i)) \end{aligned}$$

and  $\tilde{\eta}$  restricts to an endomorphism  $\tilde{\eta}^\tau = \text{Ad } \tilde{V}_1$  of  $\mathcal{A}_{\text{SU}(n),1}^{\mathbb{Z}_2} = \mathcal{A}_{\text{Spin}(n),2}$ , because  $\tilde{\eta}$  commutes with  $\sigma$  and therefore with the Jones projection on the fixpoint. In particular, we have constructed  $\tilde{V}_1 \in \mathcal{E}(\mathcal{A}_{\text{Spin}(n),2})$ .  $\square$

## 5. CONCLUSIONS AND OUTLOOK

By exploiting the explicit construction of a family of conformal nets containing loop group nets of simply laced groups at level 1, namely conformal nets associated with lattices, we have obtained semigroup elements of the Longo–Witten semigroup  $\mathcal{E}(\mathcal{A})$ . These elements give rise to new models in BQFT, i.e. local, time-translation covariant nets on Minkowski half-plane.

The level 1 loop group models can also be embedded in free Fermi nets, which could lead to different elements of the semigroup, coming from restrictions of second quantization unitaries.

It would be desirable to analyze the semigroup for loop group models at higher level. These loop group nets are subnets of the tensor product of level 1 nets and one could ask if the here obtained endomorphism restrict to these subnets. By applying the coset and orbifold construction one obtains new nets and should also get new semigroup elements. A simple example using the orbifold construction we have given in this paper.

Regarding the Longo–Witten semigroup  $\mathcal{E}(\mathcal{A})$  in general, remarkable questions and applications arise. An example is the mystery relation between elements of semigroup and integrable models with factorizing S-matrix [ZZ79] on two dimensional Minkowski space, constructed in the operator algebraic setting in [Lec08]. Both of them take inner symmetric (or scattering) functions as an input, but at the moment a deeper relation is not yet found.

Noteworthy applications of the Longo–Witten semigroup can be noticed in deformations of chiral conformal nets, where the endomorphisms  $\text{Ad } V$  associated with  $V \in \mathcal{E}(\mathcal{A})$  bring deformations of chiral CQFT's on two dimensional Minkowski space. Particularly, in [Tan11] the endomorphisms are used for a family of deformations of the  $U(1)$ -current net and the Ising net which are both second quantization nets. In this point the question that arises is if such deformations also exists for the endomorphisms of the conformal nets associated with lattices (obtained in this work), or more general for any Longo–Witten endomorphism.

Another application could be the construction of massive models in higher dimensions from conformal nets. Here the idea is, basically, that the restriction of a massive free field net to a light-ray gives a conformal net; then certain translations yield Longo–Witten endomorphisms. In [BMRW09] it is shown, in a field theoretic context, how to reconstruct the massive theory, namely how one obtains back the scalar massive free field from infinity copies of the  $U(1)$ -current. This idea translated back to the algebraic context uses one-parameter groups of the Longo–Witten semigroup. Unfortunately, the Hölder continuity rule out the functions  $\varphi_t(p) = \exp(-it/p)$  that produce a one-parameter group with negative generator needed to construct a 2D local net. We hope to come back to this issue, and possibly new constructions of nets in higher dimensions from conformal nets.

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